# Euler Classes and Frobenius algebras

by

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# Abstract

### Euler Classes and Frobenius algebras

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This thesis investigates the relationship between the handle element of the De Rham cohomology algebra of a compact oriented smooth manifold, thought of as a Frobenius algebra, and the Euler class of the manifold. In this way it gives a complete answer to an exercise posed in the monograph of Kock [5] (which is based on a paper of Abrams [6]), where one is asked to show that these two classes are equal. Firstly, an overview of De Rham cohomology, Thom and Euler classes of smooth manifolds, Poincaré duality, Frobenius algebras, and their graphical calculus is given. Finally, it is shown that the handle element and the Euler class are indeed equal for even-dimensional manifolds. However, they are not equal for odd-dimensional manifolds.

# Uittreksel


**Euler Classes and Frobenius algebras**

*("Euler classes and Frobenius algebras")*

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Hierdie proefskrif ondersoek die verband tussen die handvatselelement van die De Rham-kohomologie-algebra van 'n kompakte georiënteerde gladde variëteit, beskou as 'n Frobenius-algebra, en die Euler-klas van die variëteit. Op hierdie manier gee dit 'n volledige antwoord op 'n oefening wat in die monografie van Kock [5] gebaseer is (wat gebaseer is op 'n papier van Abrams [6]), waar een gevra word om te wys dat hierdie twee klasse gelyk is. Eerstens word 'n oorsig gegee van De Rham-kohomologie, Thom- en Euler-klasse van gladde manifolds, Frobenius algebras, en hul grafiese notasie word gegee. Ten slotte word getoon dat die handvatsel en die Euler-klas inderdaad gelyk is vie ewe-dimensionele variëteite. Maar hulle is nie gelyk nie vir onewe-dimensionele variëteite nie.

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# Dedications

*To God, dad, mom, Fy, Salomé and Bruce.*

# Contents

# Nomenclature

**Linear Algebra**

| | |
|---|---|
| $\pi^*$ | The pullback of $\pi$ |
| $\mathbb{R}^n$ | The set of all vectors of size $n$ with real entries. |
| $id$ | The identity map. |
| $\Delta$ | The diagonal map. |
| $\pi_*$ | The integration map. |
| $j_\bullet$ | The extension map. |

# Chapter 1

# Introduction

In Mathematics, classification of objects is very important in order to better understand the object itself. Regarding Algebra, the Artin-Wedderburn theorem stated below gives the classification of a finite dimensional semisimple algebra.

*Theorem(Artin-Wedderburn)*
*A finite dimensional semisimple algebra over a field $\mathbb{K}$ is isomorphic to a finite product $\Pi M_{n_i}(D_i)$ where the $n_i$ are natural numbers, the $D_i$ are finite dimensional division algebras over $\mathbb{K}$ and $M_{n_i}(D_i)$ is the algebra of $n_i \times n_i$ matrices over $D_i$.*

In this theorem semisimplicity property is one of the pillars that lead to the classification. In 2000, Abrams (**?**) discovered a way of checking the semisimplicity of a Frobenius algebra which is given by the following theorem:

*The characteristic element of a Frobenius algebra $A$ over a field of characteristic 0 $\mathbb{K}$ is unit if and only if $A$ is semisimple.*

Here what he meant by characteristic element is exactly the same as what we mean by handle element in this thesis.
Regarding Topology, the Euler class is a powerful tool to classify vector bundles over spheres. For instance, in 1956, Milnor(**?**) could prove that all the $S^3-$bundles over $S^4$ are homeomorphic to $S^7$ when the Euler class has a value of $\pm 1$.
This thesis links Algebra and Topology in showing that the diagonal class is the comultiplication of the unit in $\mathbb{R}$, which leads to the fact that the handle

element of the cohomology ring of an even dimensional compact connected orientable manifold is equal to the Euler class of that same manifold. And as a bonus, we would like to correct some statements made in Kock's book (**?**, exercise 22, page 131) and Abrams (**?**, bottom of page 4). Regarding (**?**), we point out that the handle class is equal to the Euler class only for even-dimensional manifolds, but not for odd-dimensional manifolds. Regarding (**?**), we point out that the formula on the bottom of page 4 is not quite right, namely

$$e(X) = \sum_i e_i e_i^{\#} \quad \text{should be} \quad e(X) = \sum_i e_i^{\#} e_i.$$

To better understand that, let us look at the layout of the thesis.

- The second chapter is mainly divided into two parts. The first part provides the useful tools to understand the de Rham cohomology and the second part the explanation of the de Rham cohomology itself. The tools are composed on one hand by the chain complexes and their cohomology and on the other hand by the alternating algebra. In the chain complexes and their cohomology section we are going to have an insight on the notion of chain complexes and their map, on the cohomology of chain complexes and their map, and on the short exact sequence of chain complexes and its induced long exact sequence of cohomology of chain complexes. However in the alternating algebra, we are going to introduce the alternating forms and explain the map between two vector spaces of alternating forms. About the de Rham cohomology, there are three types of it namely, the Ordinary de Rham cohomology, the compactly supported de Rham cohomology and the compact vertical de Rham cohomology. All these cohomologies started from the simplest case which is the de Rham complex for open sets in $\mathbb{R}^n$, which is expanded into the de Rham cohomology for a manifold that here we call the Ordinary de Rham cohomology. The compactly supported cohomology enables us to define the integration of differential forms on a manifold which leads us to the generalization of the Poincaré lemma. Moreover, compact vertical cohomology is needed to define the characteristic classes of a vector bundle over a manifold. Therefore in this chapter, we are going to look at the explicit construction of these cohomologies, the Poincaré lemmas and the Mayer-Vietoris sequence.

- The Chapter 3 introduces quite different concepts which are the graphical calculus and the Frobenius algebra. The graphical calculus which is a generalization of Penrose's graphical calculus(**?**) is used to represent maps. As dealing with algebraic proofs are not obvious, the graphics will enable us to prove some statements in an easier way. So the first part of this chapter will define graphical calculus and explain how it works. Although the second part discusses more about the Frobenius algebra and its handle element. Since the notion of Frobenius algebra appears in different fields like Computer Science, Mathematics, etc, there are many version of its definition. Here we are only going to look at two of them, namely the classical and categorical definitions, followed by the proof of their equivalence. Right after that we are going to give the graphical representation of the handle element for a Frobenius algebra. Finally as an example, we will compute the handle element for the cohomology ring of a manifold.

- In Chapter 4 we focus our attention on the principal problem in this thesis, namely the following exercise from Kock's book (**?**, exercise 22, page 131)

  *(C.f. Abrams [3]) If you are aquainted with cohomology ring (cf. 2.2.23). Let $X$ be a compact connected orientable manifold of dimension $r$, and put $A = H^*(X)$. Show that the Euler class (top Chern class of the tangent bundle, $c_r(TX)$) is the handle element of $A$.*

  The key to understanding the Euler class from a Frobenius algebra point of view is to translate the *Thom class* of the tangent bundle of the manifold $M$,

  $$\mathrm{Th}_{TM} \in H^m_{cv}(TM)$$

  into the *diagonal class*, namely the Poincaré dual of $M$ as a submanifold of $M \times M$,

  $$\hat{D}_M \in H^m(M \times M)$$

  This is the viewpoint of Milnor and Stasheff (**?**, Chapter 11), and we give an overview of this approach. In Section 4.1 we show that after applying certain natural isomorphisms, the Thom class is actually equal to the diagonal class. This is useful because the diagonal class fits more easily into the framework of studying the cohomology algebra $H^*(M)$ as a Frobenius algebra. Indeed, after applying the Künneth isomorphism, the diagonal class can be thought of as an element

$$D_M \in \bigoplus_{i+j=m} H^i(M) \otimes H^j(M)$$

and hence can be directly compared with the copairing element of $H^*(M)$,

$$\gamma(1) \in \bigoplus_{i+j=m} H^i(M) \otimes H^j(M).$$

We show that for even-dimensional manifolds, the diagonal class is equal to the copairing element, while for odd-dimensional manifolds, they are not equal. Since the Euler class is derived from the Thom class, this is both a small correction (in that the statement is only true for even-dimensional manifolds), as well as a refinement (in that it's not only that the Euler class is the same as the handle element, but also that the diagonal class is the same as the copairing) of the exercise from Kock's book (**?**, exercise 22, page 131). Furthermore, a clarification on the algebraic formula for the Euler class given by Abrams (**?**, bottom of page 4) is made.

In this very end, I would like to acknowledge that some of the pictures present in this thesis are taken form Kock's book (**?**).

# Chapter 2

# De Rham cohomology, compactly supported cohomology and compact vertical cohomology

In this chapter we review the three different versions of De Rham cohomology we will need in this thesis. Firstly, there is the Ordinary De Rham cohomology of manifolds which is the vector space of closed forms that are not necessarily exact. Secondly, there is the compactly supported cohomology of manifolds, where the differential forms have compact support on the manifold. Finally, there is the compact vertical cohomology of vector bundles, where the differential forms must have compact support on the fiber.
The reader is expected to be familiar with the notion of vector spaces, manifolds, smooth map between manifolds and tangent space of manifolds.
In sections **??** and **??** we will review chain complexes and their cohomology, and the alternating algebra of a vector space. The sections **??** and **??** will cover the Ordinary De Rham cohomology for open sets in $\mathbb{R}^n$ and for a manifold. The section **??** gives us more details about compactly supported cohomology. And the section **??** takes us through the compact vertical cohomology.
The materials used but not cited here may be found in Madsen (**?**).

## 2.1 Chain complexes and their cohomology

In this section we are firstly going to look at the definition of a chain complex and its cohomology. Secondly we are going to go give an overview on the definition of a map between two chain complexes and its induced map on cohomology. And finally, we are going to define the short exact sequence of chain complexes and its induced long exact sequence.
To begin with, let us see the definition of a chain complex.

**Definition 2.1.1.** *A sequence $A^* = \{A^i, d^i\}$ of vector spaces and linear maps defined as*

$$\cdots \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} A^{i+2} \to \cdots$$

*is called a chain complex if $d^{i+1} \circ d^i = 0$ for all $i$.*
*It is said to be exact if $\operatorname{Ker} d^i = \operatorname{Im} d^{i-1}$ for all $i$.*

Now, let us define the cohomology of a chain complex.

**Definition 2.1.2.** *Let $A^* = \{A^i, d^i\}$ be a chain complex. The $p-$th cohomology vector space of $A^*$ is defined as:*

$$H^p(A^*) = \frac{\operatorname{Ker} d^p}{\operatorname{Im} d^{p-1}}.$$

Given two chain complexes, one may wonder what the definition of a map between them can be. It is given the name of a chain map, and as follows is its definition.

**Definition 2.1.3.** *A chain map $f : A^* \longrightarrow B^*$ between chain complexes consists of a family $f^p : A^p \longrightarrow B^p$ of linear maps, satisfying $d_B^p \circ f^p = f^{p+1} \circ d_A^p$. And it might be viewed as the following commutative diagram:*

$$\begin{tikzcd}
\cdots \arrow[r] & A^{p-1} \arrow[r, "d_A^{p-1}"] \arrow[d, "f^{p-1}"'] & A^p \arrow[r, "d_A^p"] \arrow[d, "f^p"'] & A^{p+1} \arrow[r] \arrow[d, "f^{p+1}"'] & \cdots \\
\cdots \arrow[r] & B^{p-1} \arrow[r, "d_B^{p-1}"'] & B^p \arrow[r, "d_B^p"'] & B^{p+1} \arrow[r] & \cdots
\end{tikzcd}$$

Given two chain complexes, we can study their cohomologies. So it is interesting to know what a chain map between two chain complexes become when it comes to cohomology. It turns out that it induces a map in cohomology as well. The following lemma is taking us through that.

**Lemma 2.1.4.** *A chain map $f : A^* \longrightarrow B^*$ induces a a linear map*

$$\begin{aligned} f^* = H^*(f) : H^p(A^*) &\longrightarrow H^p(B^*) \\ a + \operatorname{Im} d_A^{p-1} &\longmapsto f(a) + \operatorname{Im} d_B^{p-1} \end{aligned}$$

*for all $p$.*

*Proof.* The proof amounts to show that $f^*$ is well-defined.
So we first need to show that if $a \in \ker d_A^p$ then $f^p(a) \in \ker d_B^p$.
Let $a \in \ker d_A^p$. Since $f$ is a chain map, we have

$$\begin{aligned} d_B^p(f^p(a)) &= f^{p+1}(d_A^p(a)) \\ &= f^{p+1}(0) \ \ ; a \in \ker d_A^p \\ &= 0 \end{aligned}$$

Therefore $f^p(a) \in \ker d_B^p$.
Let $a_1 + \operatorname{Im} d_A^{p-1} = a_2 + \operatorname{Im} d_A^{p-1}$, $a_1, a_2 \in \ker d_A^p$. This is equivalent to say that $a_1 - a_2 \in \operatorname{Im} d_A^{p-1}$ which means there exists $x \in A^{p-1}$ such that $a_1 - a_2 = d_A^{p-1}(x)$. Therefore,

$$\begin{aligned} f^p(a_1 - a_2) &= f^p(d_A^{p-1}(x)) \\ &= d_B^{p-1}(f^{p-1}(x)) \end{aligned}$$

Hence $f^p(a_1) - f^p(a_2) \in \operatorname{Im} d_B^{p-1}$ i.e $f^p(a_1) + \operatorname{Im} d_B^{p-1} = f^p(a_2) + \operatorname{Im} d_B^{p-1}$. □

Now that we have defined the map between chain complexes, let' s move to the definition of short exact sequence of chain complexes.

**Definition 2.1.5.** *A short exact sequence of chain complexes*

$$0 \longrightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \longrightarrow 0$$

*consist of chain maps $f$ and $g$ such that $0 \longrightarrow A^p \xrightarrow{f^p} B^p \xrightarrow{g^p} C^p \longrightarrow 0$ is exact for every $p$.*

As we have seen before, a chain map induces a map between the cohomology of chain complexes. And here, as we have defined the short exact sequence of chain complexes it intrigues our mind to know if it induces a short exact sequence of cohomology of chain complexes as well. In fact, that is not true. It induces an exact sequence but not a short exact sequence. The following lemma is giving us the details about that.

**Lemma 2.1.6.** *A short exact sequence of chain complexes*

$$0 \longrightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \longrightarrow 0$$

*induces an exact sequence*

$$H^p(A^*) \xrightarrow{f^*} H^p(B^*) \xrightarrow{g^*} H^p(C^*)$$

*Proof.* We need to prove that $\ker g^* = \operatorname{Im} f^*$.
Recall

$$\begin{aligned} f^* = H^*(f) : H^p(A^*) &\longrightarrow H^p(B^*) \\ a + \operatorname{Im} d_A^{p-1} &\longmapsto f(a) + \operatorname{Im} d_B^{p-1} \end{aligned}$$

$$\begin{aligned} \operatorname{Im} f^* &= \{[b] \in H^p(B^*) \mid \exists [a] \in H^p(A^*) \text{ and } [f(a)] = [b] = f^*([a])\} \\ \ker g^* &= \{[b] \in H^p(B^*) \mid g^*([b]) = 0_{C^*}\} \\ &= \{[b] \in H^p(B^*) \mid g^*([b]) \in \operatorname{Im} d_C^{p-1}\} \\ &= \{[b] \in H^p(B^*) \mid [g(b)] \in \operatorname{Im} d_C^{p-1}\} \end{aligned}$$

- $\operatorname{Im} f^* \subseteq \ker g^*$

  Let $[b] \in \operatorname{Im} f^*$ i.e. there exists $a \in \ker d_A^p$ such that $[b] = [f(a)]$. Since $g(f(a)) = 0$ (short exact sequence), we have

  $$\begin{aligned} g^*([b]) &= g^*[f(a)] \\ &= [g(f()a)] \\ &= [0] \end{aligned}$$

  Therefore, $[b] \in \ker g^*$.

- $\ker g^* \subseteq \operatorname{Im} f^*$

  The commutativity of the diagram below gives us the exactness of the short sequence.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A^{p-1} \arrow[r, "f^{p-1}"] \arrow[d, "d_A^{p-1}"'] & B^{p-1} \arrow[r, "g^{p-1}"] \arrow[d, "d_B^{p-1}"'] & C^{p-1} \arrow[r] \arrow[d, "d_C^{p-1}"'] & 0 \\
0 \arrow[r] & A^p \arrow[r, "f^p"] \arrow[d, "d_A^p"'] & B^p \arrow[r, "g^p"] \arrow[d, "d_B^p"'] & C^p \arrow[r] \arrow[d, "d_C^p"'] & 0 \\
0 \arrow[r] & A^{p+1} \arrow[r, "f^{p+1}"] \arrow[d] & B^{p+1} \arrow[r, "g^{p+1}"] \arrow[d] & C^{p+1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Let $[b] \in \ker g^*$ i.e. $[g^p(b)] = 0$ or $g^p(b) \in \operatorname{Im} d_C^{p-1}$. That means $g^p(b) = d^{p-1}(c)$, where $c \in C^{p-1}$. Since $g^{p-1}$ is surjective, there exists $b_1 \in B^{p-1}$ such that $g^{p-1}(b_1) = c$. Therefore

$$\begin{aligned} g^p(b) &= d_C^{p-1}(g^{p-1}(b_1)) \\ &= g^p(d_B^{p-1}(b_1)) \end{aligned}$$

It follows that $g^p(b - d_B^{p-1}(b_1)) = 0$ i.e. $b - d_B^{p-1}(b_1) \in \ker g^p$. So there exists $a \in A^p$ such that $b - d_B^{p-1}(b_1) = f^p(a)$. Now, we need to show that $a \in \ker d_A^p$.

We have

$$\begin{aligned} f^{p+1}(d_A^p(a)) &= d_B^p(f^p(a)) \\ &= d_B^p(b - d_B^{p-1}(b_1)) \\ &= d_B^p(b) \\ &= 0 \end{aligned}$$

Hence, $d_A^p(a) = 0$ since $f^{p+1}$ is injective. Thus we have found a cohomology class $[a] \in H^p(A)$ and

$$\begin{aligned} f^*([a]) &= [f^p(a)] \\ &= [b - d_B^{p-1}(b_1)] \\ &= [b] \end{aligned}$$

So $[b] \in \operatorname{Im} f^*$.

□

If we cannot have a short exact sequence of cohomology of chain complexes, then let us see the best we can do on it. The following definition gives the map that connects the $p-$th cohomology of the last chain complex present in the short exact sequence and the $(p+1)-$th cohomology of the first chain complex present in the short exact sequence.

**Definition 2.1.7.** *For a short exact sequence of chain complexes* $0 \longrightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \longrightarrow 0$, *let*

$$\partial^* : H^p(C^*) \longrightarrow H^{p+1}(A^*)$$

*be the linear map defined as*

$$\partial^*([c]) = [(f^{p+1})^{-1}(d_B^p((g^p)^{-1}(c)))].$$

*which is illustrated by the slanted arrow.*

$$\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A^{p-1} \arrow[r, "f^{p-1}"] \arrow[d, "d_A^{p-1}"'] & B^{p-1} \arrow[r, "g^{p-1}"] \arrow[d, "d_B^{p-1}"'] & C^{p-1} \arrow[r] \arrow[d, "d_C^{p-1}"'] & 0 \\
0 \arrow[r] & A^p \arrow[r, "f^p"] \arrow[d, "d_A^p"'] & B^p \arrow[r, "g^p"] \arrow[d, "d_B^p"'] & C^p \arrow[r] \arrow[d, "d_C^p"'] \arrow[dll] & 0 \\
0 \arrow[r] & A^{p+1} \arrow[r, "f^{p+1}"'] \arrow[d] & B^{p+1} \arrow[r, "g^{p+1}"'] \arrow[d] & C^{p+1} \arrow[r] \arrow[d] & 0 \\
& \vdots & \vdots & \vdots &
\end{tikzcd}$$

Notice that according to Madsen(**?**) p.29, Definition 4.5, this map is well-defined.

Now that we have $\partial^*$, it encourages us to seek if there is an exact sequence between $H^p(B^*), H^p(C^*)$ and $H^{p+1}(A^*)$. Indeed, there is. And this lemma gives us an insight on that.

**Lemma 2.1.8.** *The sequence* $H^p(B^*) \xrightarrow{g^*} H^p(C^*) \xrightarrow{\partial^*} H^{p+1}(A^*)$ *is exact.*

*Proof.* Recall

$$\begin{aligned}
\ker \partial^* &= \{[c] \in H^p(C^*) \mid \partial^*([c]) = 0_{H^{p+1}(A^*)}\} \\
&= \{[c] \in H^p(C^*) \mid (f^{p+1})^{-1}(d_B^p((g^p)^{-1}(c))) \in \operatorname{Im} d_A^p\} \\
\operatorname{Im} \partial^* &= \{[c] \in H^p(C^*) \mid \exists [b] \in H^p(B^*) \textit{ and } [c] = g^*([b])\} \\
&= \{[c] \in H^p(C^*) \mid \exists [b] \in H^p(B^*) \textit{ and } [c] = [g^p(b)]\}
\end{aligned}$$

- $\operatorname{Im} g^* \subseteq \ker \partial^*$

  Let $[c] \in \operatorname{Im} g^*$ i.e. there exists $[b] \in H^p(B^*)$ such that $[c] = [g^p(b)]$. And we have

  $$\begin{aligned}
  \partial^*([g^p(b)]) &= (f^{p+1})^{-1}(d_B^p((g^p)^{-1}(g^p(b)))) \\
  &= [(f^{p+1})^{-1}(d_B^p(b))] \\
  &= 0 \qquad (\text{because } b \in \ker d_B^p)
  \end{aligned}$$

  Therefore $[c] \in \ker \partial^*$.

- $\ker \partial^* \subseteq \operatorname{Im} g^*$

  Let $[c] \in \ker \partial^*$ i.e. $\partial^*([c]) = 0$ or $(f^{p+1})^{-1}(d_B^p((g^p)^{-1}(c))) \in \operatorname{Im} d_A^p$. which means $\exists a \in A^p$ such that $(f^{p+1})^{-1}(d_B^p((g^p)^{-1}(c))) = d_A^p(a)$.

  Since $g^p$ is surjective, we can choose $b \in B^p$ such that $c = g^p(b)$.

  So

  $$\begin{aligned}
  d_A^p(a) &= (f^{p+1})^{-1}(d_B^p((g^p)^{-1}(g^p(b)))) \\
  &= (f^{p+1})^{-1}(d_B^p(b))
  \end{aligned}$$

  And $d_B^p(b) = f^{p+1}(d_A^p)(a) = d_B^p(f^p(a))$. Therefore $d_B^p(b - f^p(a)) = 0$ and $g^p(b - f^p(a)) = g^p(b) - g^p(f^p(a)) = c$ since the short sequence is exact.

  Hence $g^*([b - f^p(a)]) = [c]$ i.e. $[c] \in \operatorname{Im} g^*$.

□

Now that we have the previous exact sequence, it becomes interesting to see if the sequence of $H^p(C^*), H^{p+1}(A^*)$ and $H^{p+1}(B^*)$ is exact. And it is actually true. The following lemma explains more about that.

**Lemma 2.1.9.** *The sequence* $H^p(C^*) \xrightarrow{\partial^*} H^{p+1}(A^*) \xrightarrow{f^*} H^{p+1}(B^*)$ *is exact.*

*Proof.*

- $\operatorname{Im} \partial^* \subseteq \ker f^*$

  Let $[c] \in H^p(C^*)$. So we have $\partial^*([c]) = [(f^{p+1})^{-1}(d_B^p((g^p)^{-1}(c)))]$. Since $g^p$ is surjective, there exists $b \in B^p$ such that $g^p(b) = c$.

  So

$$\begin{aligned} f^*(\partial^*([c])) &= f^*[(f^{p+1})^{-1}(d_B^p((g^p)^{-1}(c)))] \\ &= f^*[(f^{p+1})^{-1}(d_B^p((g^p)^{-1}(g^p(b))))] \\ &= f^*[(f^{p+1})^{-1}(d_B^p(b))] \\ &= [f^{p+1}(f^{p+1})^{-1}(d_B^p(b))] \\ &= [d_B^p(b)] \\ &= 0_{H^{p+1}(B^*)} \end{aligned}$$

  Therefore, $[c] \in \ker f^*$.

- $\ker f^* \subseteq \operatorname{Im} \partial^*$

  Lt $[a] \in \ker f^*$ i.e. $f^*([a]) = 0_{H^{p+1}(B^*)}$ or $[f^{p+1}(a)] = 0$, which means $f^{p+1}(a) \in \operatorname{Im} d_B^p$ and so there exists $b \in B^p$ such that $f^{p+1}(a) = d_B^p(b)$. Therefore $a = (f^{p+1})^{-1}(d_B^p(b))$. And we can write $b = (g^p)^{-1}(c)$ where $c \in C^p$, since $g^p$ is surjective. So now, we are left to prove that $c \in \ker d_C^p$.

  We have

$$\begin{aligned} d_C^p(c) &= d_C^p(g^p()b) \\ &= g^{p+1}(d_B^p(b)) \\ &= g^{p+1}(f^{p+1}(a)) \quad = 0_{C^{p+1}} \text{since the short sequence is exact} \end{aligned}$$

  Hence, $[a] = (f^{p+1})^{-1}(d_B^p((g^p)^{-1}(g^p(b)))) = \partial^*(g^p(b))$.

$\square$

In combining all those previous results, a very important theorem comes out.

**Theorem 2.1.10.** *Let* $0 \longrightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \longrightarrow 0$ *be a short exact sequence of chain complexes. Then the sequence*

$$\cdots \longrightarrow H^p(A^*) \xrightarrow{f^*} H^p(B^*) \xrightarrow{g^*} H^p(C^*) \xrightarrow{\partial^*} H^{p+1}(A^*) \xrightarrow{f^*} H^{p+1}(B^*) \longrightarrow \cdots$$

*is exact.*

## 2.2 Alternating algebra

In this section we define the alternating algebra on a vector space, the map on alternating algebras and its functorial properties.
Throughout this section, $V$ will represent a vector space over $\mathbb{R}$.
To start with let's have a look at the definition of a vector space of alternating forms and that of a graded algebra.

**Definition 2.2.1.** *Let $V$ be a vector space over $\mathbb{R}$. A $k$-linear map $\omega : V^k \longrightarrow \mathbb{R}$ is said to be alternating if $\omega(\xi_1, \cdots, \xi_k) = 0$ whenever $\xi_i = \xi_j$ for some pair $i \neq j$. The vector space of alternating, $k-$linear maps is denoted by $Alt^k(V)$. Notice that $Alt^0(V) = \mathbb{R}$.*

**Definition 2.2.2.**

- *A graded $\mathbb{R}$-algebra $A_*$ is a sequence of vector spaces $A_k, k \in \mathbb{N}$, and bilinear maps $\mu : A_k \times A_l \longrightarrow A_{k+l}$ which are associative.*
- *An algebra $A_*$ is called graded commutative, if*

$$\mu(a, b) = (-1)^{kl}\mu(b, a)$$

*for $a \in A_k$ and $b \in A_l$.*

On one hand, a $k-$form on $V$ is also said to be alternating if for all $k-$tuples of vectors in $V$ which presents one vector repeating consecutively $\omega$ vanishes on it. Furthermore, given a $k-$tuples of vectors in $V$ we obtain that $\omega$ applied to the permutation of that tuple only differs from $\omega$ applied to the original tuple by the sign of the permutation.
On the other hand, the set of all the alternating $k-$forms on $V$ is a real vector space. One might wonder if it can have a structure of an algebra. Well, the usual operation that one may apply on forms is the wedge product. And since the wedge product of two $k-$forms belongs to the vector space of $2k-$forms, it can't have a structure of algebra under that operation. However, the $Alt^*(V) = \oplus_k Alt^k(V)$ is an algebra under the wedge product. Moreover, this algebra is graded commutative.

**Remark 2.2.3.** *The algebra $Alt^*(V)$ is called the exterior or alternating algebra associated to $V$.*

When we have a set one usually wonders about what we can do with it. So he/she tries to put structures on it then start all the study. But after the structure is set up, most of the time the next step is to study the functions between two sets of the same structure. So in our case, given two vector spaces $V$ and $W$. Suppose that they are related by a function $f$. It turns out that there is a natural map relating $Alt^k(V)$ to $Alt^k(W)$ built out from $f$. And the next definition gives us more details about that.

**Definition 2.2.4.** *Let $V, W$ be vector spaces. Let $f : V \longrightarrow W$ be a linear map. This map $f$ induces a linear map*

$$Alt^k(f) : Alt^k(W) \longrightarrow Alt^k(V) \tag{2.1}$$

*where $Alt^k(f)(\omega)(\zeta_1, \cdots, \zeta_k) = \omega(f(\zeta_1), \cdots, f(\zeta_k))$.*
*And the composition works as follows: $Alt^k(f \circ g) = Alt^k(g) \circ Alt^k(f)$. Furthermore, $Alt^k(id_V) = id_{Alt^k(V)}$.*

## 2.3 The De Rham complex for open sets in $\mathbb{R}^n$

The previous section has prepared us to the construction of the De Rham complex on an open set of $\mathbb{R}^n$ by giving us the detailed overview on alternating algebra. Now, this coming section is explaining more about that construction explicitly. Moreover, this section will provide an insight on maps between two cohomology groups for two open sets of two different power of $\mathbb{R}$.
As a starter let's define the vector space of differential $k-$forms.

**Definition 2.3.1.** *Let $U$ be an open set in $\mathbb{R}^n$.*
*A differential $k-$ form on $U$ is a smooth map $\omega : U \longrightarrow Alt^k(\mathbb{R}^n)$. The vector space of such maps is called the vector space of $k-$differential forms and denoted by $\Omega^k(U)$.*
*Since $Alt^0(\mathbb{R}^n) = \mathbb{R}$ we have $\Omega^0(U) = C^\infty(U, \mathbb{R})$.*

Now that we have a smooth map, one might ask the explicit definition of its derivative. Indeed, it's defined as follows:

**Definition 2.3.2.** *Let $(e_1, \cdots, e_n)$ be the standard basis of $\mathbb{R}^n$.*
*The derivative of a smooth map $\omega : U \longrightarrow Alt^p(\mathbb{R}^n)$ at a point $x \in U$ is given by the linear map*

$$D_x\omega : \mathbb{R}^n \longrightarrow Alt^p(\mathbb{R}^n)$$

*such that*

$$D_x(e_i) = \frac{d}{dt}\omega(x + te_i)_{t=0} = \frac{\partial\omega}{\partial x_i}(x).$$

As we talk about vector spaces of differential forms, it intrigues immediately our mind what could be the exterior differentiation so that we may construct a chain complex out of it. The following definition explicates that exterior differentiation.

**Definition 2.3.3.** *Let $U$ be an open set in $\mathbb{R}^n$. The exterior differential*

$$diff : \Omega^k(U) \longrightarrow \Omega^{k+1}(U)$$

*is the linear operator defined as*

$$diff_x\omega(\zeta_1, \cdots, \zeta_{k+1}) = \Sigma_{l=1}^{k+1}(-1)^{l-1}D_x\omega(\zeta_l)(\zeta_1, \cdots, \hat{\zeta}_l, \cdots, \zeta_{k+1})$$

*where* $(\zeta_1, \cdots, \hat{\zeta}_l, \cdots, \zeta_{k+1}) = (\zeta_1, \cdots, \zeta_{l-1}, \zeta_{l+1}, \cdots, \zeta_{k+1})$ *and* $\omega \in \Omega^k(U)$.

**Remark 2.3.4.** *We may have a chain complex $\{\Omega^k(U), diff^k\}$ of differential forms on $U$ which is denoted as $\Omega^*(U)$.*

Since the chain complex of vector spaces of differential forms on an open subset of a certain power of $\mathbb{R}$ is understood, now we may talk about its cohomology.

**Definition 2.3.5.** *Let $U$ be an open set in $\mathbb{R}^n$.*
*The $k-th$ de Rham cohomology of $U$, denoted $H^k(U)$ is the $k-th$ cohomology vector space of $\Omega^*(U)$. And the issued chain complex which is the De Rham complex is denoted $H^*(U)$.*

Here again, similarly to the Definition **??**, one might ask if we may find a relation between two vector spaces of differential forms on two different subsets of two different power of $\mathbb{R}$. Actually the answer is yes and it's detailed by the following definition.

**Definition 2.3.6.** *Let $U_1 \subseteq \mathbb{R}^n$, $U_2 \subseteq \mathbb{R}^m$ be open sets and $\phi : U_1 \longrightarrow U_2$ a smooth map. This induces a morphism*

$$\Omega^k(\phi) : \Omega^k(U_2) \longrightarrow \Omega^k(U_1)$$

*defined as*

$$\Omega^k(\phi)(\omega)_x = Alt^k(D_x\phi)(\omega_{\phi(x)})$$

Since the cohomology of an open set of $\mathbb{R}^n$ is just the quotient of two subspaces of the vector space of differential forms, therefore it naturally inherits all the structures that this one has.

**Remark 2.3.7.**
- *Since* $H^k(U) = \frac{\operatorname{Ker} diff^k}{\operatorname{Im} diff^{k-1}}$ *we may define*

$$H^k(\phi) : H^k(U_2) \longrightarrow H^k(U_1)$$

*such that*

$$H^k(\phi) = [\Omega^k(\phi)(\omega)]$$

- *From now on* $\phi^*$ *will represent* $\Omega^k(\phi)$ *or* $H^k(\phi)$*. And it's called* ***pullback map****.*

Furthermore this new map has the ability to split with wedge product, to commute with the exterior differentiation and of being a contravariant functor.

**Theorem 2.3.8.** *Let* $U_1 \subseteq \mathbb{R}^n$*,* $U_2 \subseteq \mathbb{R}^m$ *be open sets and* $\phi : U_1 \longrightarrow U_2$ *a smooth map.*

*(a)* $\phi^*(\omega \wedge \tau) = \phi^*(\omega) \wedge \phi^*(\tau)$ *for* $\omega_1, \omega_2 \in H^k(U_2)$

*(b)* $\phi^*(f) = f \circ \phi$ *when* $f \in \Omega^0(U_2)$

*(c)* $diff\phi^*(\omega) = \phi^*(diff\omega)$

**Theorem 2.3.9.** *Let* $U_1 \subseteq \mathbb{R}^n$*,* $U_2 \subseteq \mathbb{R}^m, U_3 \subseteq \mathbb{R}^k$ *be open sets and let* $\phi : U_1 \longrightarrow U_2, \psi : U_2 \longrightarrow U_3$ *be smooth maps.*
*Then*

$$(\psi \circ \phi)^* = \phi^* \circ \psi^*$$

$$id^*_{U_1} = id_{H^*(U_1)}$$

To end this section, let us look at the Poincaré lemma for $\mathbb{R}^n$.

**Theorem 2.3.10** (Poincaré lemma)**.** *The de Rham cohomology for* $\mathbb{R}^n$ *is* $\mathbb{R}$ *in degree* 0 *and* 0 *elswhere,i.e.*

$$H^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & * = 0 \\ 0 & otherwise \end{cases}$$

## 2.4 The De Rham complex for a manifold

In the previous section we have encountered the De Rham complex for an open set of $\mathbb{R}^n$. But one may ask if all those results might be generalized for a manifold. We define in this section the De Rham complex for a smooth manifold, study its functorial properties, and define the Mayer-Vietoris sequence.
To start with, let's define the vector space of differential forms on a manifold.
Let $M$ be a smooth manifold of dimension $m$.
A local parametrization is the inverse of a smooth chart.
Let $\omega = \{\omega_p\}_{p\in M}$ be a family of alternating $k-$forms on $T_pM$ i.e. $\omega_p \in Alt^k(T_pM)$.
Let $W$ be an open subset of $\mathbb{R}^m$ and $g : W \longrightarrow M$ be a local parametrization.
Let $x \in W$.
The map

$$D_xg : \mathbb{R}^m \longrightarrow T_{g(x)}M$$

is an isomorphism, therefore it induces an isomorphism

$$Alt^k(D_xg) : Alt^k(T_{g(x)}M) \longrightarrow Alt^k(\mathbb{R}^m)$$

defined as $Alt^k(D_xg)(\omega)(\zeta_1, \cdots, \zeta_k) = \omega(D_xg(\zeta_1), \cdots, D_xg(\zeta_k))$.
Consider the map $g^*(\omega) : W \longrightarrow Alt^k(\mathbb{R}^m)$ whose value at $x$ is

$$g^*(\omega)_x = Alt^k(D_xg)(\omega_{g(x)})$$

**Definition 2.4.1.** *Let $M$ be a smooth manifold of dimension $m$.*
*Let $W$ be an open subset of $\mathbb{R}^m$.*
*A family $\omega = \{\omega_p\}_{p\in M}$ of alternating $k-$forms on $T_pM$ is said to be smooth if $g^*(\omega) : W \longrightarrow Alt^k(\mathbb{R}^m)$ is a smooth function for every local parametrization $(W, g)$. The set of such smooth families is a real vector space $\Omega^k(M)$.*

And again, we must give the explicit definition of the exterior differentiation in order to get a chain complex of vector spaces of differential form on $M$. So as follows is the definition.

**Definition 2.4.2.** *Let $M$ be a smooth manifold of dimension $m$. The exterior differentiation is given by:*

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

*such that* $d_p\omega = Alt^{k+1}((D_x g)^{-1})(diff_x(g^*\omega))$ *where* $p = g(x)$.

**Remark 2.4.3.**
- *Note that* $\Omega^0(M) = C^\infty(M, \mathbb{R})$.
- *Similarly to the Remark* **??** *we have a chain complex* $\{\Omega^k(M), d^k\}$ *of differential forms on* $M$ *denoted* $\Omega^*(M)$.

Now, the de Rham cohomology may be defined.

**Definition 2.4.4.** *The* $k-$*th de Rham cohomology of the manifold* $M$*, denoted* $H^k(M)$ *is the* $k-$*th cohomology vector space of* $\Omega^*(M)$*. And its De Rham complex* $\{H^k(M), d^k\}$ *is denoted* $H^*(M)$.

In the previous section, we have defined the map between the $k-$th cohomology of two open subsets of two different power of $\mathbb{R}$. And we called it **pullback map** in Remark **??**. Here, we have that same concept for a map between two smooth manifolds. As follows is its definition.

**Definition 2.4.5.** *Let* $M, N$ *be smooth manifolds and* $\phi : M \longrightarrow N$ *be a smooth map. It induces a chain map*

$$\phi^* : \Omega^*(N) \longrightarrow \Omega^*(M)$$

*such that if* $\tau \in \Omega^k(N)$*, and* $p \in M$ *we have:*

$$\phi^*(\tau)_p = Alt^k(D_p\phi)(\tau_{\phi(p)})$$

*where* $\tau_{\phi(p)}$ *lives in* $Alt^k(T_{\phi(p)}N)$.

From $\phi^*$ we may derive a linear map $H^k(\phi) : H^k(N) \longrightarrow H^k(M)$ such that $H^k(\phi)[(\tau)_p] = [\phi^*(\tau)_p]$ for $p \in M$. We will still denote $H^k(\phi)$ as $\phi^*$.
And it turns out that all the results of the Theorem **??** and **??** still hold.

Now suppose that $M$ is the union of two open sets namely $U_1$ and $U_2$. Then with the inclusions $j_\nu$ and $i_\nu$ for $\nu = 1, 2$, we obtain two sequences given by

$$U_1 \cap U_2 \xrightarrow{j_1} U_1 \xrightarrow{i_1} M$$

$$U_1 \cap U_2 \xrightarrow{j_2} U_2 \xrightarrow{i_2} M$$

**Theorem 2.4.6.** *The sequence*

$$0 \longrightarrow \Omega^p(M) \xrightarrow{I^p} \Omega^p(U_1) \oplus \Omega^p(U_2) \xrightarrow{J^p} \Omega^p(U_1 \cap U_2) \longrightarrow 0$$

*is exact. Where* $I^p(\omega) = (i_1^*(\omega), i_2^*(\omega))$, $J^p(\omega_1, \omega_2) = j_1^*(\omega_1) - j_2^*(\omega_2)$.

And it follows from the Theorem **??** that this short exact sequence induces a long exact sequence for cohomology.

**Theorem 2.4.7** (Mayer-Vietoris)**.** *The Mayer-Vietoris sequence*

$$\cdots \longrightarrow H^p(M) \xrightarrow{I^*} H^p(U_1) \oplus H^p(U_2) \xrightarrow{J^*} H^p(U_1 \cap U_2) \xrightarrow{\partial^*} H^{p+1}(U) \longrightarrow \cdots$$

*is exact.*
*Notice that* $I^*([\omega]) = (i_1^*([\omega]), i_2^*([\omega]))$, $J^*([\omega_1], [\omega_2]) = j_1^*([\omega_1]) - j_2^*([\omega_2])$, *and*

$$\partial^*([\tau]) = \begin{cases} [-d(\rho_{U_2}\tau)] & \text{on } U_1 \\ [d(\rho_{U_1}\tau)] & \text{on } U_2 \end{cases}$$

*where* $\{\rho_{U_1}, \rho_{U_2}\}$ *is a partition of unity subordinate to the open cover* $\{U_1, U_2\}$ *and* $\tau \in H^p(U_1 \cap U_2)$.

## 2.5 Compactly supported cohomology and Poincaré lemma

Until now, we have encountered the de Rham cohomology of a manifold without any condition. But there is a second type of de Rham cohomology recall **compactly supported cohomology** which is slightly different from the previous one. This section will take us through this new cohomology and drive us to the generalized Poincaré lemma.
Before everything, let us first define the support of a differential form, and what it means to be a compactly supported form.

**Definition 2.5.1.** *(**?**) Let $M$ be a smooth manifold.*
*Let* $\omega \in \Omega^p(M)$.
*The support of $\omega$ is defined by* $supp(\omega) = \overline{\{p \in M, \omega_p \neq 0\}}$.

**Definition 2.5.2.** *(**?**) Let $M$ be a smooth manifold.*
*Let $\omega \in \Omega^p(M)$.*
*The differential form $\omega$ is called compactly supported if its support is compact in $M$.*

The following remark is stating that all the results that we obtained in the previous section still hold for compactly supported cohomology.

**Remark 2.5.3.**

- *The vector space of compactly supported differential $k-$forms on $M$ is denoted $\Omega_c^k(M)$. Its definition is exactly the same as in the previous section, but the only difference is that all the forms are compactly supported. Furthermore, even the exterior differentiation remains the same. And similarly to the Remark **??** we may have the chain complex of compactly supported differential forms on $M$ as well, which we denote $\Omega_c^*(M) = \{\Omega_c^p(M), d^p\}$.*

  *Hence, as in the Definition **??** we have a compactly supported de Rham cohomology $H_c^p(M)$ which is the $p-$th cohomology vector space of $\Omega_c^*(M)$.*

- *However, we should be aware that if $U \subset M$ is open, a form $\omega$ with compact support on $M$ will not restrict to a form with compact support in $U$ unless the inclusion map from $U$ to $M$ is proper.*

Although, there is a new operation that we may do on compactly supported differential forms, which is the integration.
Let $M$ be a smooth orientable manifold of dimension $m$.
Let $\pi : M \times \mathbb{R} \longrightarrow M$.
Our aim is to construct a map $\pi_* : \Omega_c^*(M \times \mathbb{R}) \longrightarrow \Omega_c^{*-1}(M)$ which will give rise to a map $\pi_* : H_c^*(M \times \mathbb{R}) \longrightarrow H_c^{*-1}(M)$ which we call **integration map** over the fiber.
Let $(U, \varphi)$ be a chart around $p \in M$ and $x = (x_1, \cdots, x_m)$ be the local coordinate at $p$. So we get $(U \times \mathbb{R}, \varphi \times id)$ as a chart for $M \times \mathbb{R}$.
If $\omega = \Sigma_I f_I(x) dx_I$ is a compactly supported differential form on $U$ then a $k-$form on $U \times \mathbb{R}$ looks like

$$\Sigma_{|I|=k} \alpha_I(x_1, \cdots, x_m, t) d\tilde{x}_I + \Sigma_{|J|=k-1} \beta_J(x_1, \cdots, x_m, t) d\tilde{x}_J \wedge dt$$

where

$$\begin{aligned} d\tilde{x}_I &= d\tilde{x}_{i_1} \wedge \cdots \wedge d\tilde{x}_{i_k} \\ &= d(\pi^* x_{i_1}) \wedge \cdots \wedge d(\pi^* x_{i_k}) \\ &= \pi^*(dx_{i_1}) \wedge \cdots \wedge \pi^*(dx_{i_k}) \\ &= \pi^*(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \\ &= \pi^*(dx_I) \end{aligned}$$

and $\alpha_I, \beta_J$ are smooth. In addition, $\beta_J$ is integrable over $\mathbb{R}$ for all $J$.
Let our $\pi_*$ be defined as follows:

(i) $\alpha_I(x_1, \cdots, x_m, t)d\tilde{x}_I \longmapsto 0$

(ii) $\beta_J(x_1, \cdots, x_m, t)d\tilde{x}_J \wedge dt \longmapsto (\int_{-\infty}^{\infty} \beta_J(x_1, \cdots, x_m, t)dt)dx_J$

One of the properties of the integration map is given by the following theorem.

**Theorem 2.5.4** (Generalized Poincaré lemma). *(**?**) The integration map*

$$\pi_* : H_c^*(M \times \mathbb{R}) \longrightarrow H_c^{*-1}(M)$$

*is an isomorphism.*

*Proof.* See Bott and Tu p.39 □

In particular we have the Poincaré lemma for the compactly supported cohomology for $\mathbb{R}^n$.

**Lemma 2.5.5** (Poincaré lemma). *The compactly supported de Rham cohomology of $\mathbb{R}^n$ is equal to 0 in dimension $n$, and 0 elswhere, i.e.*

$$H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & * = n \\ 0 & otherwise \end{cases}$$

Similarly to the previous section, we have Mayer-Vietoris sequence for compactly supported forms. However it is slightly different. So let us use a different notation.
Let $V \subset U$ be open subsets of $M$, and let $i$ be the inclusion map $i : V \longrightarrow U$. Then $i$ induces a chain map

$$\begin{aligned} i_\bullet : \Omega_c^*(V) &\longrightarrow \Omega_c^*(U) \\ \omega \longmapsto i_\bullet(\omega) &= \begin{cases} \omega & \text{on } V \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

Notice that the extension of the composition of two maps is just the composition of the extension of the two maps.
So suppose that $M$ is the union of two open sets namely $U_1$ and $U_2$. Then with the inclusions $j_\nu$ and $i_\nu$ for $\nu = 1, 2$, we obtain two sequences given by

$$U_1 \cap U_2 \xrightarrow{j_1} U_1 \xrightarrow{i_1} M$$

$$U_1 \cap U_2 \xrightarrow{j_2} U_2 \xrightarrow{i_2} M$$

**Theorem 2.5.6.** *(**?**) The sequence*

$$0 \longrightarrow \Omega_c^p(U_1 \cap U_2) \xrightarrow{I_p} \Omega_c^p(U_1) \oplus \Omega_c^p(U_2) \xrightarrow{J_p} \Omega^p(M) \longrightarrow 0$$

*is exact. Where* $I_p(\omega_1, \omega_2) = (i_1)_\bullet(\omega) + (i_2)_\bullet(\omega))$ *and* $J_p(\omega) = ((j_1)_\bullet(\omega), -(j_2)_\bullet(\omega))$.

And it follows from the Theorem **??** that this short exact sequence induces a long exact sequence for cohomology.

**Theorem 2.5.7** (Mayer-Vietoris compact)**.** *(**?**) The Mayer-Vietoris sequence*

$$\cdots \longrightarrow H_c^p(U_1 \cap U_2) \xrightarrow{I_\bullet} H_c^p(U_1) \oplus H_c^p(U_2) \xrightarrow{J_\bullet} H_c^p(M) \xrightarrow{\partial_\bullet} H_c^{p+1}(U_1 \cap U_2) \longrightarrow \cdots$$

*is exact.*
*Notice that* $I_\bullet([\omega_1], [\omega_2]) = (i_1)_\bullet([\omega_1]) + (i_2)_\bullet([\omega_2]))$ *and* $J_\bullet([\omega]) = ((j_1)_\bullet([\omega_1]), -(j_2)_\bullet([\omega_2]))$.

## 2.6 Compact vertical cohomology

Apart from compactly supported de Rham cohomology mentioned before, there exists another kind of cohomology that we may apply on a vector bundle over a manifold which is called **compact vertical cohomology** or **compactly supported cohomology in the vertical direction**. This section gives us an overview on vector bundles, the definition of a compact vertical cohomology and the generalization of the integration map.
So let's start with the definition of a vector bundle.

**Definition 2.6.1.** *A rank n vector bundle* $\xi = (E, B, \pi_E)$ *consists of :*

1. *a smooth manifold E, called the total space*
2. *a smooth manifold B, called the base space*

3. *a smooth map* $\pi_E : E \longrightarrow B$

4. *for each* $x \in B$*, there is a structure of a vector space on* $E_x := \pi_E^{-1}(x)$

   *such that for each* $x \in B$ *there exists an open neighborhood* $U \subset B$ *of* $x$ *and a diffeomorphism*
   $$h : U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U)$$
   *where, for each* $x \in U$*, the map*
   $$\mathbb{R}^n \longrightarrow E_x$$
   *which sends* $v \in \mathbb{R}^n$ *to* $h(x, v)$ *is a linear isomorphism.*

   *The map* $h$ *is called trivialization.*

Here are some important examples of vector bundles that we are going to use all the way along.

**Example 2.6.2.** 1. *Let* $M$ *be a manifold. Set* $TM = \{(p, v) : p \in M, v \in T_p(M)\}$*. The tangent bundle over* $M$ *is a vector bundle where* $E = TM$ *which we may denote as* $(TM, M, \pi_{TM})$*. And* $\pi_{TM} : TM \longrightarrow M$ *defined as* $\pi_{TM}((p, v)) = p$*.*

2. *Let* $M$ *be a manifold endowed with a Riemannian metric and* $S$ *be a submanifold of* $M$*.*

   *Let* $N_p(S) = (T_p(S))^{\perp}$ *where* $T_p(S)$ *denotes the tangent space of* $S$ *at the point* $p$*.*

   *Set* $N(S) = \cup_{p \in M} N_p(S)$*. Then* $(N(S), S, \pi_{N(S)})$ *is a vector bundle over* $S$*. The projection* $\pi_{N(S)}$ *is such that* $\pi_{N(S)} : N(S) \longrightarrow S$ *defined as* $\pi_{N(S)}((p, v)) = p$ *for* $v \in N_p(S)$*.*

Now, let's define the compact vertical vector space of a vector bundle over a smooth oriented manifold.

**Definition 2.6.3.** *(**?**) Let* $M$ *be a smooth oriented manifold and* $(E, M, \pi_E)$ *be a vector bundle over* $M$*.*
$\Omega^k_{cv}(E)$ *is the vector space of differential* $k-$*forms on* $E$ *with compact support in the vertical direction i.e. the forms do not need to have compact support in* $E$ *but its restriction to each fiber has compact support.*

**Remark 2.6.4.** *As we have seen in the two previous sections, we may have a chain complex of vector spaces of vertical compactly supported differential forms on E which we denote $\{\Omega^*_{cv}(E), d\}$ since E itself is a manifold. And the de Rham complex $H^*_{cv}(E)$ follows.*

We may generalize the concept of integration map in Chapter **??** by the integration map over the fiber that we will still call **integration map**.

**Definition 2.6.5.** *Let M be an $m-$dimensional smooth oriented manifold. Let $(E, M, \pi_E)$ be a rank n oriented vector bundle over M. Our aim is to define the integration along the fiber*

$$(\pi_E)_* : \Omega^*_{cv}(E) \longrightarrow \Omega^{*-n}(M).$$

*Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ be an oriented trivialization of E. Let $x = (x_1, \cdots, x_m)$ be the coordinate system on $U_\alpha$ and $t_1, \cdots, t_n$ be the fiber coordinates on $E|_{U_\alpha}$. Let's define $(\pi_E)_*$ as :*

*(I) all the compactly supported on the vertical direction differential forms on $U_\alpha \times \mathbb{R}^n$ of the form $\Sigma_I f_I(x, t_, \cdots, t_n) dx_I \wedge dt_{j_1} \cdots \wedge dt_{j_r}$ maps to 0 when $r < n$,*

*(II) all the compactly supported on the vertical direction differential forms on $U_\alpha \times \mathbb{R}^n$ of the form $\Sigma_J f_J(x, t_, \cdots, t_n) dx_J \wedge dt_1 \wedge \cdots dt_n$ maps to $\Sigma_J (\int_{\mathbb{R}^n} f_J(x, t_1, \cdots, t_n) dt_1 \wedge \cdots \wedge dt_n) dx_J$.*

And, $(\pi_E)_*$ satisfies a formula called Projection formula.

**Proposition 2.6.6** (Projection formula)**.** *Let M be a smooth oriented manifold.*

*a) Let $(E, M, \pi_E)$ be an oriented vector bundle of rank n over M, $\omega \in \Omega^q_{cv}(E)$ and $\tau \in \Omega^{m+n-q}(M)$ with compact support along the fiber. Then*

$$(\pi_E)_*(\pi_E^* \tau \wedge \omega) = \tau \wedge (\pi_E)_* \omega.$$

*b) As we consider M as oriented of dimension m, if $\omega \in \Omega^q_{cv}(E)$ and $\tau \in \Omega^{m+n-q}(M)$, then*

$$\int_E \pi_E^* \tau \wedge \omega = \int_M \tau \wedge (\pi_E)_* \omega.$$

*Proof.* See Bott and Tu p.63 □

# Chapter 3

# Frobenius algebra and its handle element

The mathematician Richard Brauer and his PhD student Cecil James Nesbitt began to study Frobenius algebras in 1930. However, the interest in it has currently been renewed due to its connections to Topological Quantum Field Theory as shown in Robbert Dijkgraaf's PhD thesis in 1989.
A Frobenius algebra is an algebra defined by a form called Frobenius form. And its operator called handle operator which is defined by a specific element called handle element is very important for the algebraic topologist, because it turns out that it represents a hole in the topological point of view. So our aim in this Chapter is to compute the handle element of a Frobenius algebra in the topological and algebraic way. Therefore, to start with, the first section will take us through the techniques to represent an algebraic entity into a topological one. Then, the following section will take us through the definitions of a Frobenius algebra. Next, the third section will give us more details about the handle operator and the handle element. Eventually, the last section explains about the Frobenius algebra structure on the cohomology ring of a connected compact orientable smooth manifold and the algebraic computation of its handle element.
Notice that throughout this chapter, the following maps will remain the same:

- the multiplication $\mu : A \otimes A \longrightarrow A$
- the comultiplication $\delta : A \longrightarrow A \otimes A$

- the Frobenius form $\epsilon : A \longrightarrow \mathbb{K}$
- the unit $\eta : \mathbb{K} \longrightarrow A$
- the pairing $\beta : A \otimes A \longrightarrow \mathbb{K}$
- the copairing $\gamma : \mathbb{K} \longrightarrow A \otimes A$

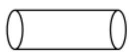
## 3.1 Graphical calculus

The physicist and mathematician Roger Penrose, in 1971, has introduced a new method of representing maps called **graphical calculus** or **string diagrams** (**?**). As every new tool needs a manual, here it is:

- For each $\mathbb{K}-$linear map $\phi : A^m \longrightarrow A^n$ $(A^m = \underbrace{A \otimes \cdots \otimes A}_{\text{m times}})$,we represent it by a particular cobordism where $m$ are the in-boundaries and $n$ the out-boundaries of the cobordism.
- If we have a tensor products of $k$ linear maps, we draw them in parallel vertically (from top to bottom) by reversing the order of each factor (i.e. the last factor goes to the top and the first factor to the bottom).
- For the composition $f \circ g$, we glue the out-boundaries of $g$ with the in-boundaries of $f$.

**Example 3.1.1.** 1. *The maps defining a $\mathbb{K}-$algebra are given by:*

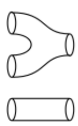
(a) unit $\eta$



(b) identity $id_A$

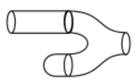
(c) multiplication $\mu$

2. *The tensor product $id_A \otimes \mu$*



3. *The composition*

$$\mathbb{K} \otimes A \xrightarrow{\eta \otimes id_A} A \otimes A \xrightarrow{\mu} A$$



## 3.2 Frobenius algebra

As we want to connect Algebra and Topology, there are two definitions of Frobenius algebra that we will be giving in this section, namely the classical and the categorical one that here we will call Frobenius definition. And of course the last part of the section will show the equivalence between those definitions.

### 3.2.1 Frobenius algebra in the classical point of view

**Definition 3.2.1** (classical)**.** *A Frobenius algebra is a $\mathbb{K}$-algebra $A$ equipped with a linear functional $\epsilon : A \longrightarrow \mathbb{K}$ whose associated pairing $\beta(a,b) = \epsilon(ab)$ is nondegenerate.*

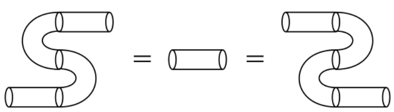
Note: The form $\epsilon$ is called the Frobenius form.

**Definition 3.2.2.** *(**?**) Let $A$ be a $\mathbb{K}-$algebra. A pairing $\beta : A \otimes A \longrightarrow \mathbb{K}$ is called left (right) nondegenerate if there exists a copairing $\gamma : \mathbb{K} \longrightarrow A \otimes A$ such that the left (right) diagram commute:*

$$\begin{tikzcd} A \arrow[r, "\gamma\otimes id_A"] \arrow[dr, "id_A"'] & A\otimes A\otimes A \arrow[d, "id_A\otimes\beta"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "id_A\otimes\gamma"] \arrow[dr, "id_A"'] & A\otimes A\otimes A \arrow[d, "\beta\otimes id_A"] \\ & A \end{tikzcd}$$

*And the pairing is called nondegenerate if it is both left and right nondegenerate.*

By using the graphical representation, nondegeneracy is given by a relation called **snake relation** where the "left snake" represents the left nondegeneracy and the "right snake" represents the right nondegeneracy.



However, there is the algebraic definition of the nondegeneracy of a pairing as well, which is given by the following theorem.

**Theorem 3.2.3.** *(?) A paring is left nondegenerate if and only if the map*

$$\begin{aligned} \beta_{left} : A &\longrightarrow A^* \\ a &\longmapsto \beta(-, a) \end{aligned}$$

*is an isomorphism.*
*And similarly, it is right nondegenerate if and only if the map*

$$\begin{aligned} \beta_{right} : A &\longrightarrow A^* \\ a &\longmapsto \beta(a, -) \end{aligned}$$

*is an isomorphism*

And this theorem brings us to a lemma.

**Lemma 3.2.4.** *Let $\{b_1, \cdots, b_n\}$ be a basis for a Frobenius algebra $A$. Then there exists a unique dual basis $\{b_1^{\#}, \cdots, b_n^{\#}\}$ of $A$ such that $\beta(b_i, b_i^{\#}) = \delta_{ij}$.*

*Proof.*

- To start with let's prove the existence of $\{b_1^{\#}, \cdots, b_n^{\#}\}$.

  Since $\{b_1, \cdots, b_n\}$ is a basis of $A \otimes A$ and $\gamma(1_{\mathbb{K}}) \in A \otimes A$, then we have

  $$\begin{aligned} \gamma(1_{\mathbb{K}}) &= \sum_{i,j} a_{ij} b_i \otimes b_j \\ &= \sum_j (\sum_i a_{ij} b_i) \otimes b_j \\ &= \sum_j c_j \otimes b_j \end{aligned}$$

  So define, $b_j^{\#} = c_j$. And now we may write $\gamma(1_{\mathbb{K}}) = \sum_j b_j^{\#} \otimes b_j$ for some elements $b_1^{\#}, \cdots, b_n^{\#} \in A$.

- The second thing to do is to show that $\beta(b_i, b_j^{\#}) = \delta_{ij}$. From the snake relation we have

  $$\begin{aligned} A \otimes \mathbb{K} &\xrightarrow{id \otimes \gamma} A \otimes A \otimes A \xrightarrow{\beta \otimes id} \mathbb{K} \otimes A \\ b_i \otimes 1_{\mathbb{K}} &\longmapsto \sum_i b_i \otimes b_j^{\#} \otimes b_j \longmapsto \sum_j \beta(b_i, b_j^{\#}) \otimes b_j = b_i \end{aligned}$$

  And from the fact that $\{b_1, \cdots, b_n\}$ is a basis, we have $\beta(b_i, b_j^{\#}) = \delta_{ij}$

- Thirdly, let's prove the linearly independence of $b_1^\#, \cdots, b_n^\#$.

  Suppose that $\sum_{i=1}^n a_i b_i^\# = 0$ for $a_i \in \mathbb{K}$. Then we have

$$\beta(b_j, \sum_{i=1}^n a_i b_i^\#) = 0$$

  which is equivalent to say

$$\sum_{i=1}^n a_i \underbrace{\beta(b_j, b_i^\#)}_{\delta_{ij}} = 0$$

  Therefore, $a_i = 0$ for $i = 1, \cdots, n$.

  Furthermore $b_1^\#, \cdots, b_n^\#$ span $A$ since they are $n$ linearly independent vectors in an $n-$dimensional vector space.

- Finally, let's prove the uniqueness of $b_1^\#, \cdots, b_n^\#$.

  Suppose there exists $(b_1^\#)', \cdots, (b_n^\#)'$ such that

$$\beta(b_i, b_j^\#) = \delta_{ij} \; \forall i, j \in \{1, \cdots, n\}.$$

  Then

$$\beta(b_i, b_j^\# - (b_j^\#)') = 0 \; \forall i, j \in \{1, \cdots, n\}. \tag{3.1}$$

  Define $v_j = b_j^\# - (b_j^\#)'$, so we have $\beta(b_i, v_j) = 0 \; \forall i, j \in \{1, \cdots, n\}$.

  By using the snake relation

$$\begin{array}{ccccc} \mathbb{K} \otimes A & \xrightarrow{\gamma \otimes id} & A \otimes A \otimes A & \xrightarrow{id \otimes \beta} & A \otimes \mathbb{K} \\ 1_{\mathbb{K}} \otimes v & \longmapsto & \sum_j b_j^\# \otimes b_j \otimes v & \longmapsto & b_j^\# \otimes \sum_{j=1}^n \beta(b_j, v) = v \end{array}$$

  which may be interpreted as

$$\sum_{j=1}^n \beta(b_j, v) b_j^\# = v$$

  In particular, when $v = v_j$, we obtain $v_j = 0 \; \forall j \in \{1, \cdots, n\}$ from (**??**). Which means $b_j^\# = (b_j^\#)'$.

□

**Lemma 3.2.5.** *The copairing of a Frobenius algebra can be expressed as*

$$\gamma(1_{\mathbb{K}}) = \sum_i b_i^{\#} \otimes b_i \tag{3.2}$$

*where $\{b_i\}_i$ is any basis of $A$ and $\{b_i^{\#}\}_i$ is the dual basis.*

*Proof.* In fact, what we need to do here is to check if the snake relation holds.

$$\begin{array}{ccccc} \mathbb{K} \otimes A & \xrightarrow{\gamma \otimes id} & A \otimes A \otimes A & \xrightarrow{id \otimes \beta} & A \otimes \mathbb{K} \\ 1_{\mathbb{K}} \otimes b_i^{\#} & \longmapsto & \sum_i b_i^{\#} \otimes b_i \otimes b_i^{\#} & \longmapsto & b_i^{\#} \otimes 1_{\mathbb{K}} = b_i^{\#} \end{array}$$

$$\begin{array}{ccccc} A \otimes \mathbb{K} & \xrightarrow{id \otimes \gamma} & A \otimes A \otimes A & \xrightarrow{\beta \otimes id} & \mathbb{K} \otimes A \\ b_i \otimes 1_{\mathbb{K}} & \longmapsto & \sum_i b_i \otimes b_i^{\#} \otimes b_i & \longmapsto & 1_{\mathbb{K}} \otimes b_i = b_i \end{array}$$

□

### 3.2.2 Frobenius algebra in a categorical point of view

The classical definition of a Frobenius algebra may be viewed in a categorical point of view which we call Frobenius definition. In this subsection we are mainly working on that definition. This new definition involves algebras and coalgebras, so we are going to define them in an appropriate way.

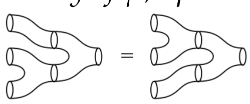
**Definition 3.2.6** (algebra). *A $\mathbb{K}-$algebra is a $\mathbb{K}-$vector space $A$, together with two $\mathbb{K}-$linear maps*

$$\mu : A \otimes A \longrightarrow A, \ \eta : \mathbb{K} \longrightarrow A$$

*such that the three following diagrams, which are respectively called associativity and unit conditions, commute :*

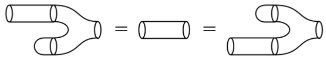
$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "id_A \otimes \mu"] \arrow[d, "\mu \otimes id_A"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd}$$

*Associativity of $\mu$,represented as*



$$\begin{tikzcd} \mathbb{K} \otimes A \arrow[r, "\eta\otimes id_A"] \arrow[d] & A \otimes A \arrow[dl, "\mu"] \\ A & \end{tikzcd} \qquad \begin{tikzcd} A \otimes \mathbb{K} \arrow[r, "id_A\otimes\eta"] \arrow[d] & A \otimes A \arrow[dl, "\mu"] \\ A & \end{tikzcd}$$

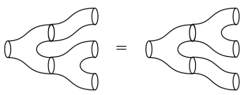
*unit conditions, represented as*



**Definition 3.2.7** (Coalgebra)**.** *A coalgebra over $\mathbb{K}$ is a vector space $A$ together with two $\mathbb{K}-$linear maps $\delta : A \longrightarrow A \otimes A$, $\epsilon : A \longrightarrow \mathbb{K}$ such that the three following diagrams, which are called respectively coassociativity and counit condition, commute:*

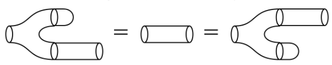
$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "\delta"'] & A \otimes A \arrow[d, "id_A\otimes\delta"] \\ A \otimes A \arrow[r, "\delta\otimes id_A"'] & A \otimes A \otimes A \end{tikzcd}$$

*Coassociativity of $\delta$, graphically represented as*



$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr, "\delta"'] & A \otimes A \arrow[d, "\epsilon\otimes id_A"] \\ & \mathbb{K} \otimes A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr, "\delta"'] & A \otimes A \arrow[d, "id_A\otimes\epsilon"] \\ & A \otimes \mathbb{K} \end{tikzcd}$$

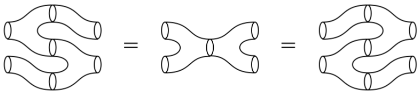
*Counit axioms, graphically represented as*



**Definition 3.2.8 (Frobenius definition).** *A Frobenius algebra is a vector space equipped with the structure of algebra and coalgebra, satisfying the Frobenius relation:*

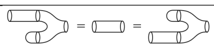
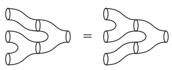
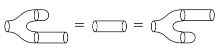
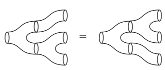
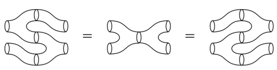
$$\begin{tikzcd}
 & A \otimes A \arrow[dl, "\delta\otimes id_A"'] \arrow[d, "\mu"] \arrow[dr, "id_A\otimes\delta"] & \\
A \otimes A \otimes A \arrow[dr, "id_A\otimes\mu"'] & A \arrow[d, "\delta"] & A \otimes A \otimes A \arrow[dl, "\mu\otimes id_A"] \\
 & A \otimes A &
\end{tikzcd}$$

*Frobenius relation, represented as*

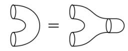
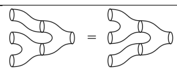
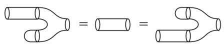
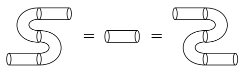


### 3.2.3 Equivalence between the two definitions of Frobenius algebra

In fact, what we would like to do here is to summarise both definitions in two different tables by using the graphical meaning, and then try to fill in the missing picture in each one of them that will make them equal . So the Frobenius definition may be summarised as:

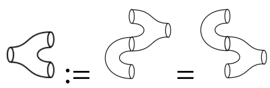
| **Frobenius definition of a Frobenius algebra** | |
|---|---|
| **Pictures** | **Algebraic meaning** |
|  | multiplication |
|  | unit |
|  | comultiplication |
|  | counit |
|  | the unit satisfies the unit axiom |
|  | the multiplication is associative |
|  | the counit satisfies the counit axiom |
|  | The comultiplication is coassociative |
|  | the Frobenius relation holds |

And the summary of the Classical definition goes like:

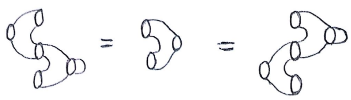
| **Picturised details of the classical definition of a Frobenius algebra** | |
|---|---|
| **Classical definition** | **Graphical representation** |
| $\mu : A \otimes A \longrightarrow A$ |  |
| $\eta : \mathbb{K} \longrightarrow A$ |  |
| $\epsilon : A \longrightarrow \mathbb{K}$ |  |
| pairing $\beta : A \otimes A \longrightarrow \mathbb{K}$ |  |
| copairing $\gamma : \mathbb{K} \longrightarrow A \otimes A$ |  |
| Associativity of $\mu$ |  |
| Counit axiom |  |
| nondegeneracy of $\beta$ |  |

**Theorem 3.2.9.** *The Classical definition is equivalent to the Frobenius definition.*

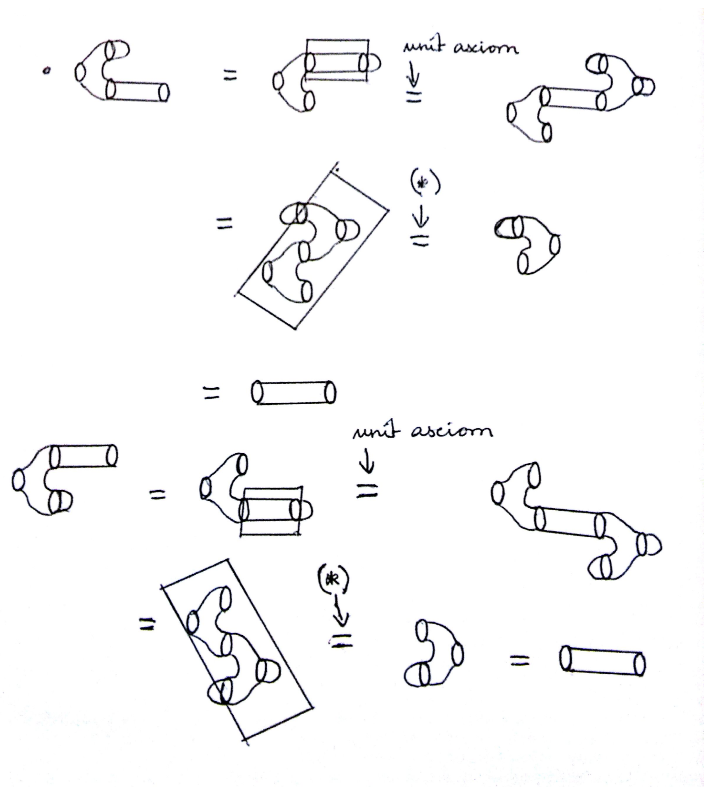
*Proof.* ($\Longrightarrow$)On one hand, let's suppose that we have the Classical definition. From those tables, we can see that the grapical datas in Frobenius definition appear in the Classical definition. However the comultiplication is missing. But, since we have the mutiplication and the copairing, the comultiplication might be defined as follows :



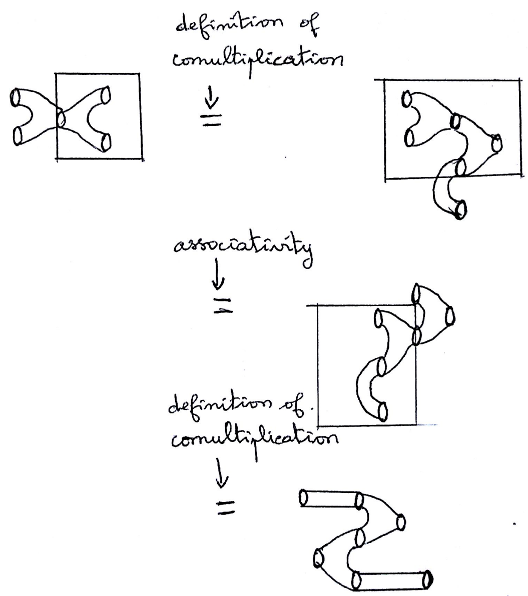
Furthermore, in the relations section the counit axiom, the Frobenius relation and the coassociativity of the comultiplication are missing. So firstly, let us prove that the counit axiom is satisfied in the Classical definition.
Notice that the instruction given above the sign "=" means that we apply this relation on the figure put inside rectangle.
So now that we have the comultiplication, from the Lemma 2.3.17(**?**) the multiplication can be expressed in terms of the comultiplication as follows:

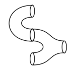
(*) 

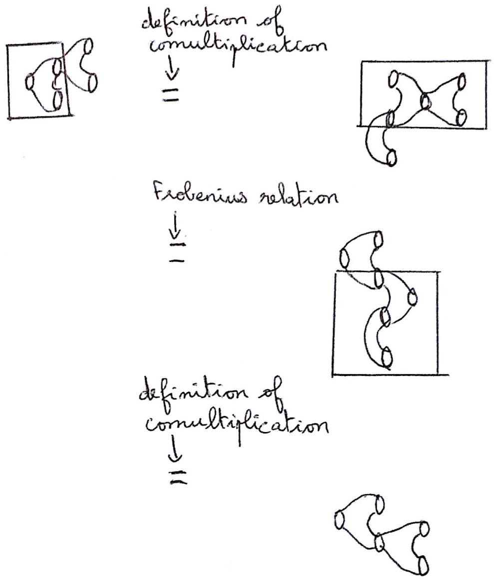
And the proof of the counit axiom is given as follows:

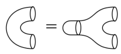


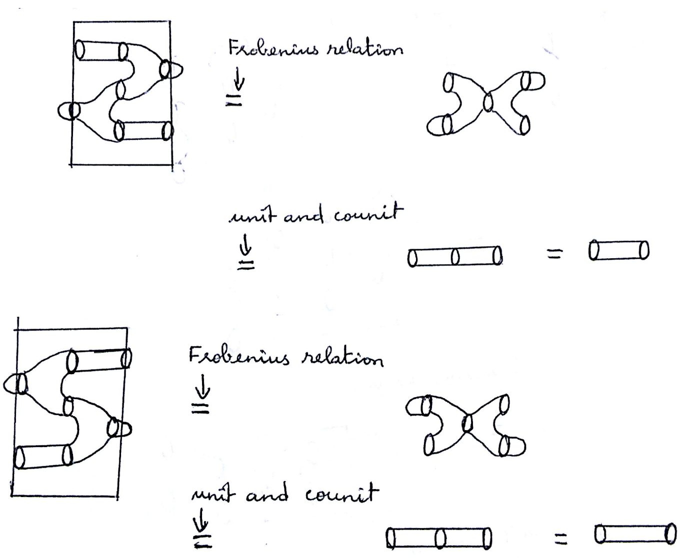

Hence, we have the counit axiom.
Secondly, let' s show that all we have in the classical definition is enough for the Frobenius relation to hold. In fact, the associativity of the multiplication and the existence of the comultiplication gives us the Frobenius relation as follows:



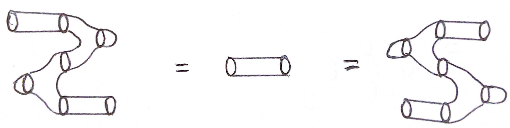
and the other equality is obtained by the same process by using the other definition of comultiplication which is .
Finally, since we have the Frobenius relation we may have the coassociativity of the comultiplication as well as follows:




($\Longleftarrow$)
On the other hand, suppose that we have the Frobenius definition.
We may consider the counit as the Frobenius form. And from the data that we have, we may obtain the pairing associated to $\epsilon$ and the copairing might be considered as . So the only thing left to prove is the snake relation which represents the nondegeneracy of the pairing.
We have



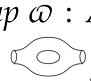

Therefore



Thus, indeed the two definitions are equivalent. □

## 3.3 Handle operator and handle element

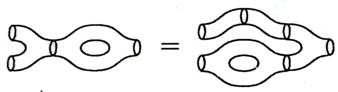
Given an algebra, an operator might be defined on it. In this section we are particularly interested in the handle operator of a Frobenius algebra.

**Definition 3.3.1** (Handle operator)**.** *(?) Let $A$ be a $\mathbb{K}-$algebra. The handle operator is the $\mathbb{K}-$linear map $\varpi : A \longrightarrow A$ defined as the composite $\mu \circ \delta$ which is graphically represented by* .

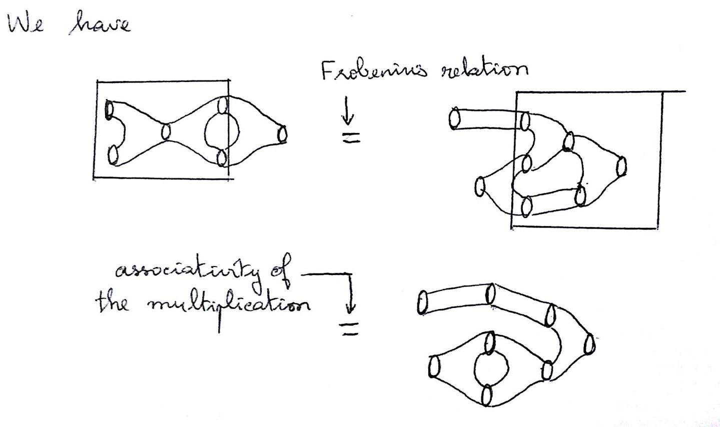
For a Frobenius algebra, a handle operator has the property of being a module homomorphism which is explained in the following theorem.

**Theorem 3.3.2.** *(?) If $A$ is a Frobenius algebra then $\varpi : A \longrightarrow A$ is a right (and left) $A-$module homomorphism, i.e. the following diagram commutes:*

$$\begin{tikzcd} A \otimes A \arrow[r, "\varpi \otimes id_A"] \arrow[d, "\mu"] & A \otimes A \arrow[d, "\mu"] \\ A \arrow[r, "\varpi"'] & A \end{tikzcd}$$

*and which we may represent as* 

*Proof.* Note: The text on top of the equality sign is meant to be applied on the figure inside the rectangle.




□

We may say that the previous definition of a handle operator is topological. So it is a little bit difficult to use it in calculation. Therefore, here is the algebraic one.

**Definition 3.3.3.** (**?**) *Let $A$ be a $\mathbb{K}-$algebra. An element $u$ in $A$ is said to be central if $xu = ux$ for all $x$ in $A$.*

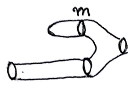
**Theorem 3.3.4.** *Let $A$ be a Frobenius algebra. The handle operator is given by multiplication by a central element which is called the* ***handle element****.*

*Proof.* Our aim here is to construct explicitly the graphical representative of the handle element.
Therefore, for a better understanding, the first step is to represent the statement "$\forall a \in A, ma = am$" graphically.

Let $M : \mathbb{K} \longrightarrow A$ such that $1 \longmapsto m$.
We may have $ma$ as follows:

$$\mathbb{K} \otimes A \xrightarrow{M \otimes id_A} A \otimes A \xrightarrow{\mu} A$$

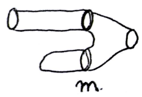
$$1 \otimes a \longmapsto m \otimes a \longmapsto ma$$



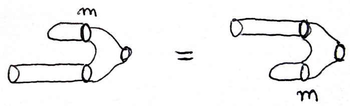

Similarly, we may have $am$ as follows:

$$A \otimes \mathbb{K} \xrightarrow{id_A \otimes M} A \otimes A \xrightarrow{\mu} A$$

$$a \otimes 1 \longmapsto a \otimes m \longmapsto am$$

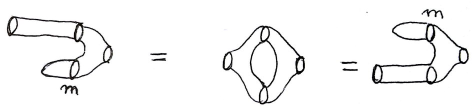



Therefore we have

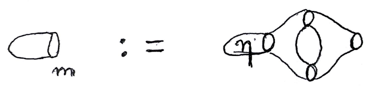



Our goal is to find

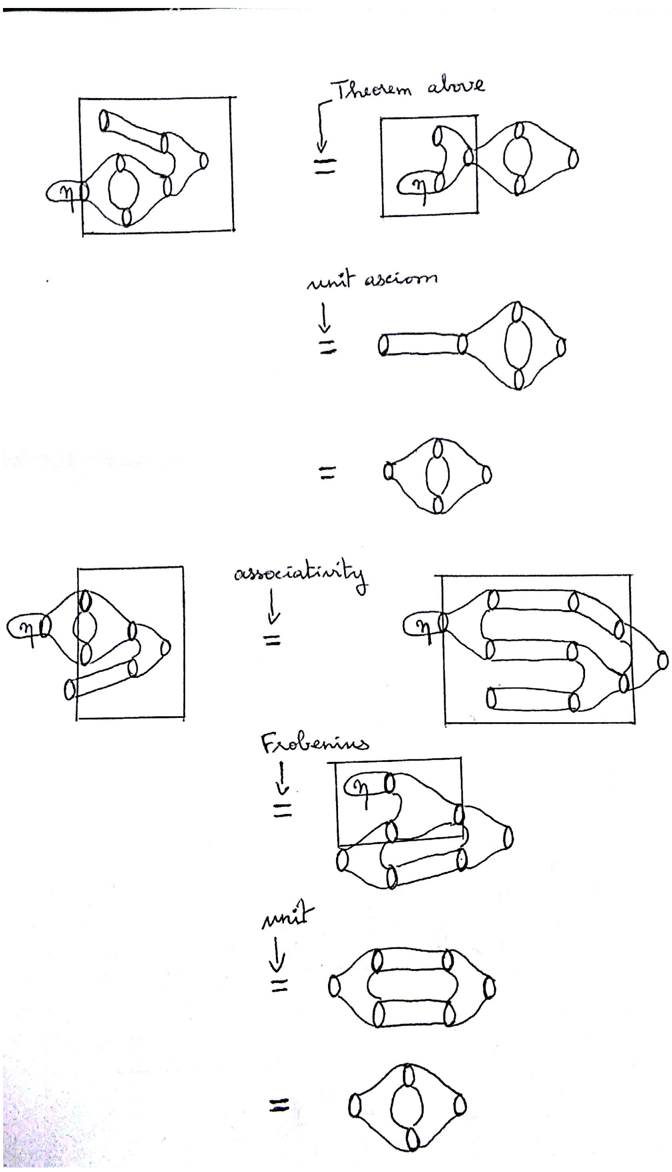

such that




By taking




It follows that




□

Remember that the main significance of the handle operator is given by the following theorem:

**Theorem 3.3.5** (Abrams)**.** *(**?**) If* $\mathbb{K}$ *is algebraically closed, then*

$$A \text{ semisimple} \Longleftrightarrow \text{the handle element } m \text{ is invertible.}$$

## 3.4 Example of Frobenius algebra: $H^*(M)$, and its handle element

Let $M$ be a compact oriented smooth manifold of even dimension $m$.
The previous sections enlightened us about the general concept of a Frobenius algebra and its handle operator. But, this coming section provides a particular example of Frobenius algebra which is $H^*(M)$ and its handle element. To be precise, we will give the explicit definitions of the multiplication $\mu$, the unit $\eta$, the Frobenius form $\epsilon$ and its corresponding pairing $\beta$. Right after that we are going to see the definition of the copairing and eventually, give the algebraic formula for the handle element of $H^*(M)$.
Note that $H^*(M)$ is an $\mathbb{R}-$algebra with multiplication given by

$$\begin{aligned} \mu : H^*(M) \otimes H^*(M) &\longrightarrow H^*(M) \\ \omega_1 \otimes \omega_2 &\longmapsto \omega_1 \wedge \omega_2 \end{aligned}$$

and a unit $1 \in H^0(M)$ which is the constant function.
The Frobenius form is given by

$$\begin{aligned} \epsilon : H^*(M) &\longrightarrow \mathbb{R} \\ \alpha &\longmapsto \int_M \alpha \end{aligned}$$

And its corresponding pairing $\beta$ by:

$$\begin{aligned} \beta : H^*(M) \otimes H^*(M) &\longrightarrow \mathbb{R} \\ \omega_1 \otimes \omega_2 &\longmapsto \int_M \omega_1 \wedge \omega_2 \end{aligned}$$

To show the nondegeneracy of the pairing $\beta$ we make use of the Poncaré duality theorem stated below and its simplified version for compact manifolds.
The Poincaré duality theorem and Theorem **??** implies that the pairing $\beta$ is nondegenerate.
The Poincaré duality theorem is stated as follows:

**Theorem 3.4.1** (Poincaré duality). **(?)** *Let $M$ be an $m-$dimensional oriented manifold with finite good cover. Then the pairing*

$$\beta : H^q(M) \otimes H_c^{m-q}(M) \longrightarrow \mathbb{R}$$

*defined as $\beta(\omega, \alpha) = \int_M \omega \wedge \alpha$ for $\omega \in H^q(M), \alpha \in H_c^{m-q}(M)$ is nondegenerate.*

The proof of this theorem necessitates two lemmas which are stated as follows:

**Lemma 3.4.2** (Five lemma)**.** *(**?**) Consider a commutative diagram of vector spaces and linear maps with exact rows*

$$\begin{tikzcd} A_1 \arrow[r, "g_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "g_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "g_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "g_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\ B_1 \arrow[r, "h_1"'] & B_2 \arrow[r, "h_2"'] & B_3 \arrow[r, "h_3"'] & B_4 \arrow[r, "h_4"'] & B_5 \end{tikzcd}$$

*Suppose that $f_1, f_2, f_4, f_5$ are isomorphisms. Then so is $f_3$.*

*Proof.* The commutativity of the diagrams amounts to say the following results:

$$\begin{array}{lll} \operatorname{Im} g_1 = \ker g_2 & \operatorname{Im} h_1 = \ker h_2 & f_2 \circ g_1 = h_1 \circ f_1 \\ \operatorname{Im} g_2 = \ker g_3 & \operatorname{Im} h_2 = \ker h_3 & f_3 \circ g_2 = h_2 \circ f_2 \\ \operatorname{Im} g_3 = \ker g_4 & \operatorname{Im} h_3 = \ker h_4 & f_4 \circ g_3 = h_3 \circ f_3 \\ & & f_5 \circ g_4 = h_4 \circ f_4 \end{array}$$

- Let's show that $f_3$ is injective.

  Let $a_3 \in A_3$ such that $f_3(a_3) = 0$. So $h_3(f_3(a_3)) = 0$ i.e. $f_4(g_3(a_3)) = 0$. Since $f_4$ is injective, we have $g_3(a_3) = 0$ i.e. $a_3 \in \ker g_3 = \operatorname{Im} g_2$. Therefore there exists $a_2 \in A_2$ such that $a_3 = g_2(a_2)$. From the assumption that $f_3(a_3) = 0$ and by substituting $a_3$ by $g_2(a_2)$ we have $f_3(g_2(a_2)) = 0 = h_2(f_2(a_2))$ i.e. $f_2(a_2) \in \ker h_2 = \operatorname{Im} h_1$. Hence, there exists $b_1 \in B_1$ such that $f_2(a_2) = h_1(b_1)$. But $f_1$ is surjective so there exists $a_1 \in A_1$ such that $b_1 = f_1(a_1)$. Therefore

  $$\begin{aligned} f_2(a_2) &= h_1(f_1(a_1)) \\ &= (h_1 \circ f_1)(a_1) \\ &= (f_2 \circ g_1)(a_1) \\ &= f_2(g_1(a_1)) \end{aligned}$$

  Thus $a_2 = g_1(a_1)$ since $f_2$ is injective i.e. $a_2 \in \operatorname{Im} g_1 = \ker g_2$. Whence $a_3 = g_2(a_2) = 0$ and $f_3$ is injective.

- Now, let's show that $f_3$ is surjective i.e. for all $b_3 \in B_3$ there exists $a_3 \in A_3$ such that $b_3 = f_3(a_3)$.

Let $b_3 \in B_3$. Since $f_4$ is surjective, there exists $a_4 \in A_4$ such that $f_4(a_4) = h_3(b_3)$. From that we have

$$\begin{aligned} h_4(h_3(b_3)) &= 0 \\ &= h_4(f_4(a_4)) \\ &= f_5(g_4(a_4)) \end{aligned}$$

So $g_4(a_4) = 0$ since $f_5$ is injective, i.e. $a_4 \in \ker g_4 = \operatorname{Im} g_3$. Therefore there exists $a_3 \in A_3$ such that $a_4 = g_3(a_3)$. And we obtain $f_4(a_4) = f_4(g_3(a_3)) = h_3(b_3)$ i.e. $h_3(f_3(a_3)) = h_3(b_3)$, which means $h_3(f_3(a_3) - b_3) = 0$. But that is equivalent to say that there exists $b_2 \in B_2$ such that $f_3(a_3) - b_3 = h_2(b_2)$ which leads us to $b_3 = f_3(a_3) - h_2(b_2)$. As we know that $f_2$ is surjective, then there exists $a_2 \in A_2$ such that $b_2 = f_2(a_2)$. Thus

$$\begin{aligned} b_3 &= -h_2(f_2(a_2)) + f_3(a_3) \\ &= -f_3(g_2(a_2)) + f_3(a_3) \\ &= f_3(a_3 - g_2(a_2)) \end{aligned}$$

And $f_3$ is surjective.

□

**Lemma 3.4.3.** *(?) By combining the two Mayer-Vietoris sequences Theorem* **??** *and Theorem* **??**
*Recall*

$$\cdots \longrightarrow H^p(M) \xrightarrow{I^*} H^p(U_1) \oplus H^p(U_2) \xrightarrow{J^*} H^p(U_1 \cap U_2) \xrightarrow{\partial^*} H^{p+1}(M) \longrightarrow \cdots$$

$$\cdots \longleftarrow H_c^{n-p}(M) \xleftarrow{I_\bullet} H_c^{n-p}(U_1) \oplus H_c^{n-p}(U_2) \xleftarrow{J_\bullet} H_c^{n-p}(U_1 \cap U_2) \xleftarrow{\partial_\bullet} H_c^{n-p-1}(M) \longleftarrow \cdots$$

*we have the commutativity of the following diagrams:*

$$\begin{tikzcd} H^p(U_1 \cap U_2) \otimes H_c^{n-p-1}(U_1 \cup U_2) \arrow[r, "id\otimes\partial_\bullet"] \arrow[d, "\partial^*\otimes id"'] & H^p(U_1 \cap U_2) \otimes H_c^{n-p}(U_1 \cap U_2) \arrow[d, "\int_{U_1\cap U_2}"] \\ H^{p+1}(U_1 \cup U_2) \otimes H_c^{n-p-1}(U_1 \cup U_2) \arrow[r, "\int_{U_1\cup U_2}"'] & \mathbb{R} \end{tikzcd}$$

$$\begin{tikzcd}
(H^p(U_1) \oplus H^p(U_2)) \otimes H_c^{n-p}(U_1 \cap U_2) \arrow[r, "id\otimes J_\bullet"] \arrow[d, "J^*\otimes id"'] & (H^p(U_1) \oplus H^p(U_2)) \otimes (H_c^{n-p}(U_1) \oplus H^{n-p}(U_2)) \arrow[d, "\int_{U_1} + \int_{U_2}"] \\
H^p(U_1 \cap U_2) \otimes H_c^{n-p}(U_1 \cup U_2) \arrow[r, "\int_{U_1 \cap U_2}"'] & \mathbb{R}
\end{tikzcd}$$

$$\begin{tikzcd}
H^p(U_1 \cup U_2) \otimes (H_c^{n-p}(U_1) \oplus H^{n-p}(U_2)) \arrow[r, "id\otimes I_\bullet"] \arrow[d, "I^*\otimes id"'] & H^p(U_1 \cup U_2) \otimes H_c^{n-p}(U_1 \cup U_2) \arrow[d, "\int_{U_1 \cap U_2}"] \\
(H^p(U_1) \oplus H^p(U_2)) \otimes (H_c^{n-p}(U_1) \oplus H^{n-p}(U_2)) \arrow[r, "\int_{U_1} + \int_{U_2}"'] & \mathbb{R}
\end{tikzcd}$$

*And that induces the sign-commutativity of the following diagram:*

$$\begin{tikzcd}
\cdots \arrow[r] & H^p \arrow[r] \arrow[d] & H^p \oplus H^p \arrow[r] \arrow[d] & H^p \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & (H_c^{n-p})^* \arrow[r] & (H_c^{n-p})^* \oplus (H_c^{n-p})^* \arrow[r] & (H_c^{n-p})^* \arrow[r] & \cdots
\end{tikzcd}$$

*Proof.* (Poincaré duality) The fact that $M$ is compact implies that it has a good cover. And $\{U, V\}$ is a good cover for $M$ means that $U \cap V$ is contractible. We are going to prove this theorem by induction on the cardinality of the good cover of $M$.

- Let $M$ be a good cover for $M$, i.e. $M$ is contractible and $M$ is diffeomorphic to $\mathbb{R}^n$. The Poincaré lemmas **??**, **??** tell us that $H^0(\mathbb{R}^n) = \mathbb{R} = H_c^n(\mathbb{R}^n) = (H^0(\mathbb{R}^n))^*$.

- Suppose that the Poincaré duality holds for any manifold having a good cover at most $p$ open sets. Let $M$ be a compact oriented manifold and $\{U_0, \cdots, U_p\}$ be its good cover. Then $(U_0 \cup \cdots \cup U_{p-1}) \cap U_p$ has $\{U_{0,p}, \cdots, U_{p-1,p}\}$ as its good cover. By hypothesis Poincaré duality holds for $U_0 \cup \cdots \cup U_{p-1}$, $U_p$ and $(U_0 \cup \cdots \cup U_{p-1}) \cap U_p$. So from the Five lemma and the lemma **??**, it holds for $U_0 \cup \cdots \cup U_{p-1} \cup U_p$ as well, and so for any compact oriented manifold.

□

**Remark 3.4.4.** *The Theorem* **??** *amounts to say that the linear map*

$$\beta_{left} : H^q(M) \longrightarrow (H_c^{n-q}(M))^*$$

*where* $\omega \longmapsto \int_M \omega \wedge \cdot$ *is an isomorphism.*
*Notice that when* $M$ *is compact,* $H_c^q(M) = H^q(M)\ \forall q$*. Therefore, we may conclude that* $H^*(M)$ *is a Frobenius algebra.*

From its definition, nondegeneracy of $\beta$ is equivalent to the existence of $\gamma : \mathbb{R} \longrightarrow H^*(M) \otimes H^*(M)$ such that the snake relation is satisfied. Indeed, from Lemma 3.2.5 we see that the copairing is given by:

$$\begin{aligned} \gamma : \mathbb{R} &\longrightarrow H^*(M) \otimes H^*(M) \\ 1_{\mathbb{R}} &\longmapsto \sum_i b_i^{\#} \otimes b_i \end{aligned}$$

Therefore, the algebraic expression of the handle element of $H^*(M)$ is given by

$$\begin{aligned} (\mu \circ \gamma)(1_{\mathbb{R}}) &= \sum_i b_i^{\#} \wedge b_i \\ &= \sum_i (-1)^{\deg b_i \deg b_i^{\#}} b_i \wedge b_i^{\#} \end{aligned}$$

# Chapter 4

# Algebraic representation of the Euler class of a manifold

According to wikipedia, the Euler class of a vector bundle of a manifold measures how "twisted" it is. Furthermore, it is a generalization of the Euler characteristic and even considered to be the archetype for other characteristic classes of vector bundles. In this chapter, we are going to look at all the steps to obtain the algebraic formula for the Euler class of a connected compact orientable smooth manifold $M$ of finite dimension. We will conclude that it is equal to the handle element of the cohomology ring of $M$ when the dimension of $M$ is even, and not equal when the dimension of $M$ is odd. We claim that the Euler class of $M$ is none but the image of the diagonal class by the multiplication $\mu$. So the first section will take us through the construction of the diagonal class $D_M$ from the Thom class of the tangent bundle of a connected compact oriented smooth manifold $M$ of dimension $m$ equipped with a Riemannian metric and its algebraic formula, plus the fact that the diagonal class of $M$ is equal to the image of the unit of $\mathbb{R}$ under the copairing $\gamma$. And the second section explains more about how we obtained the algebraic formula of the Euler class of $TM$ by using the diagonal class plus the fact that it is equal to the handle element of the cohomology ring of $M$ when the dimension of $M$ is even. To support that statement, the third section will be showing the computation of some surfaces. Lastly, the fourth section is about some corrections and clarifications to the statements made by Kock (**?**, pg 131, exercise 22) and by Abrams (**?**, bottom of page 4).

Notice that throughout this chapter, we refer to $\Delta$ as

$$\Delta : M \longrightarrow M \times M$$

such that $\Delta(x) = (x, x)$ for all $x \in M$.
And $D_M$ will be the diagonal class.

## 4.1 The diagonal class

The diagonal class is basically the equivalent of the Thom class of the tangent bundle $TM$ of $M$, which lives in the $m-$th cohomology of $M \times M$, when $m$ is the dimension of a connected compact oriented smooth manifold $M$.

### 4.1.1 Thom isomorphism and Thom class

As the construction of the diagonal class requires the notion of Thom class, this subsection is telling us more about it. In fact, the Thom class is the inverse image of the constant function of the $0-$th cohomology of $M$ by an isomorphism called Thom isomorphism; let us first look at that isomorphism.

**Theorem 4.1.1** (Thom isomorphism theorem)**.** *(**?**) Let $M$ be an orientable smooth manifold of finite type.*
*Let $(E, M, \pi_E)$ be a vector bundle of rank $n$ over $M$.*
*Let $(\pi_E)_* : \Omega^*_{cv}(E) \longrightarrow \Omega^{*-n}(M)$ be the integration along the fiber.*
*Then,*

$$(\pi_E)_* : H^*_{cv}(E) \longrightarrow H^{*-n}(M)$$

*is an isomorphism.*

*Proof.* Let $U$ and $V$ be open subsets of $M$. Then the following sequence is exact.

$$0 \longrightarrow \Omega^p_{cv}(E|_{U\cup V}) \xrightarrow{(I_E)^p} \Omega^p_{cv}(E|_U) \oplus \Omega^p_{cv}(E|_V) \xrightarrow{(J_E)^p} \Omega^p_{cv}(E|_{U\cap V}) \longrightarrow 0$$

which is defined as exactly in Theorem **??**. Therefore this sequence is exact. And it induces a long exact sequence as in Theorem **??**. Furthermore, we have the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^*_{cv}(E|_{U\cup V}) \arrow[r, "(I_E)^*"] \arrow[d, "(\pi_E)_*"] & H^*_{cv}(E|_U)\oplus H^*_{cv}(E|_V) \arrow[r, "(J_E)^*"] \arrow[d, "(\pi_E)_*"] & H^*_{cv}(E|_{U\cap V}) \arrow[r, "\partial^*_E"] \arrow[d, "(\pi_E)_*"] & H^{*+1}_{cv}(E|_{U\cup V}) \arrow[r] \arrow[d, "(\pi_E)_*"] & \cdots \\
\cdots \arrow[r] & H^{*-n}_{cv}(U\cup V) \arrow[r, "I^*"'] & H^{*-n}_{cv}(U)\oplus H^{*-n}_{cv}(V) \arrow[r, "J^*"'] & H^{*-n}_{cv}(U\cap V) \arrow[r, "\partial^*"'] & H^{*+1-n}(U\cup V) \arrow[r] & \cdots
\end{tikzcd}$$

And this diagram is commutative. In fact for the first two diagrams, since $I^*_E, J^*_E, I^*$ and $J^*$ are almost just inclusions we have the commutativity. So let us focus more on the commutativity of the third diagram. From the Theorem **??**, we had the explicit formula for $\partial^*$ by

$$\partial^*([\omega]) = \begin{cases} [-d(\rho_V\tau)] & on\ U \\ [d(\rho_U\tau)] & on\ V \end{cases}$$

where $\omega \in H^{*-n}_{cv}(U\cap V)$.

But now we want to write the formula for $\partial^*_E$ based on that of $\partial^*$. Indeed, it is given by

$$\partial^*_E([\tau]) = \begin{cases} [-(\pi^*_E d\rho_V)\wedge\tau] & on\ E|_U \\ [(\pi^*_E d\rho_U)\wedge\tau] & on\ E|_V \end{cases}$$

On $E|_{U\cap V}$, $-\pi^*_E d\rho_V\wedge\tau = \pi^*_E d\rho_U\wedge\tau$ for $\tau\in H^*_{cv}(E|_{U\cap V})$. So the commutativity of the diagram resumes to the fact that

$$\begin{aligned}
(\pi_E)_*\partial^*_E\tau &= (\pi_E)_*(\pi^*_E d\rho_U)\wedge\tau) \\
&= (d\rho_U)\wedge(\pi_E)_*\tau \text{ from the projection formula } \mathbf{??} \\
&= \partial^*(\pi_E)_*\tau
\end{aligned}$$

Hence the diagram is commutative.

Now let us prove the theorem by induction.

- Let $U = M$ be contractible, therefore it is isomorphic to $\mathbb{R}^m$. So its vector bundle $E|_M$ is trivial, and the Thom isomorphism theorem reduces to the Generalized Poincaré lemma Theorem**??**. Therefore $(\pi_E)_*$ is an isomorphism. Thus, from the Five lemma the Thom isomorphism holds.

- The rest of the proof goes verbatim as in the proof of the Poincaré duality Theorem **??** based on the cardinality of the good cover of $M$.

□

Now, let us look at the definition of the Thom class itself.

**Definition 4.1.2** (Thom class)**.** *Let $M$ be an orientable smooth manifold of finite type.*
*Let $(E, M, \pi_E)$ be a vector bundle of rank $n$ over $M$.*
*Let $(\pi_E)_* : \Omega^*_{cv}(E) \longrightarrow \Omega^{*-n}(M)$ be the integration along the fiber.*
*Let $\pi_E^* : H^*(M) \longrightarrow H^*_{cv}(E)$ be the pullback map of $\pi_E : E \longrightarrow M$.*
*The inverse image $Th_E$ of $1_{H^0(M)}$ in $H^m(E)$ by $(\pi_E)_*$ is called* ***Thom class of the bundle*** *$(E, M, \pi_E)$. That is*

$$Th_E = (\pi_E)_*^{-1}(1_{H^0(M)}).$$

In the next few subsections we will find a concrete formula for the Thom class in terms of the Diagonal class of $M$, which is easier to compute.

### 4.1.2 Fiber bundles

Let $M$ be a connected compact orientable manifold.
The next step in the construction of the diagonal class $D_M$ requires two bundle isomorphisms: the isomorphism between the normal bundle of $\Delta(M)$ and the tangent bundle of $M$, and that of the normal bundle of a compact submanifold of $M$ and its tubular neighborhood. Therefore, in this subsection we are firstly going to look at the notion of fiber bundle and isomorphism between fiber bundles. Then explain the bundle isomorphisms cited earlier.
To start with, let us look at a concept of fiber bundle.

**Definition 4.1.3.** *(**?**)*

- *A fiber bundle over a manifold consists of three manifolds $E$ (total space),$B$ (base space), $F$ (typical fiber) and a continuous map $\pi_E : E \longrightarrow B$ noted $(E, B, F, \pi_E)$, such that for each open neighborhood $U_b$ of $b \in B$ there is a homeomorphism*

  $$h : U_b \times F \longrightarrow \pi_E^{-1}(U_b)$$

  *where $\pi_E \circ h = \pi_1$ and $\pi_1$ is the first projection.*

  *The pre-image $\pi_E^{-1}(x)$ denoted by $F_x$ is called the fiber over $x$ for $x \in B$.*

- *A rank of a fiber bundle is the dimension of $F$.*

Now, let us go through the definition of a bundle isomorphism.

**Definition 4.1.4.** *(**?**) Let $(E, M, F, \pi_E)$ and $(E', N, F', \pi_{E'})$ be fiber bundles over manifolds M and N respectively.*
*Then a diffeomorphism map $\varphi : E \longrightarrow E'$ is called a bundle isomorphism from E to $E'$ if there is a diffeomorphism $f : M \longrightarrow N$ such that the following diagram commutes:*

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\pi_E"'] & E' \arrow[d, "\pi_{E'}"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Since vector bundles are particular cases of fiber bundles, the definition of vector bundle isomorphism is almost the same as above with the extra condition: for all $x \in M$, $\varphi_x : E_x \longrightarrow E'_{f(x)}$ such that $\varphi_x(v) = \varphi(v)$ for $v \in E_x$ is an isomorphism. In the Example **??** we have seen two instances of vector bundles namely $(N(M), \Delta(M), \pi_{N(M)})$ and $(TM, M, \pi_{TM})$. And actually they are isomorphic.

**Theorem 4.1.5.** *(**?**)*
*Let $(N(M), \Delta(M), \pi_{N(M)})$ be the normal bundle over $\Delta(M)$, and $(TM, M, \pi_{TM})$ be the tangent bundle over M.*
*The map*

$$\begin{aligned} \lambda : TM &\longrightarrow N(M) \\ (p, v_p) &\longmapsto ((p, p), (v_p, -v_p)) \end{aligned}$$

*is an isomorphism of vector bundles. Here $p \in M, v_p \in T_p(M)$*
*Therefore $(N(M), \Delta(M), \pi_{N(M)}) \overset{\lambda}{\cong} (TM, M, \pi_{TM})$ since $\Delta(M) \cong M$.*

*Proof.* (Theorem **??**)
By looking at the map, we may see immediately that it is an isomorphism. However, it is not clear that it is well-defined. So this proof is showing that $\lambda$ is indeed well-defined which basically means that $N(M) \subset T(M \times M)|_{\Delta(M)}$ and that $(v_p, v_p) \in N_{(p,p)M}$
By convention we write $N(M) = N(\Delta(M))$. And by definition,

$$N(M) = \cup_{p \in M} N_{(p,p)}(M)$$

where

$$N_{(p,p)}(M) = (T_{(p,p)}\Delta(M))^{\perp}$$

And from Bott and Tu(**?**) p.66, we have

$$N(M) \oplus T\Delta(M) = T(M \times M)|_{\Delta(M)}$$

Therefore $N(M) \subset T(M \times M)|_{\Delta(M)}$.
Since

$$T_{(p,p)}(M \times M) = T_p(M) \oplus T_p(M)$$

we have

$$T_{(p,p)}\Delta(M) = \{(v_p, v_p) : v \in T_pM\}$$

So

$$N_{(p,p)}(M) = \{(v_p, -v_p) : v \in T_pM\}$$

$\square$

There is one interesting example of fiber bundle which is not a vector bundle, the tubular neighborhood of a compact submanifold of a compact manifold. It is particular because it is isomorphic to the normal bundle that same submanifold. The following Theorem and Lemma explain more about that fact.

**Theorem 4.1.6** (Tubular Neighborhood Theorem)**.** *(**?**) Let $M$ be a compact manifold of dimension $m$ equipped with a Riemannian metric. Let $S \subset M$ be a compact submanifold of dimension $s$ of $M$. Then $S$ has an open neighborhood $T$ in $M$ such that $T \overset{\kappa}{\cong} N(S)$ where $N(S)$ is the total space of the normal bundle of $S$.*

**Lemma 4.1.7.** *By taking the $T$ mentioned above and equip it with the commutative diagram below, we see that $(T, S, \mathbb{R}^{m-s}, \pi_T)$ is a fiber bundle which we will call **tubular neighborhood** over $S$ and simultaneously we have the bundle isomorphism $(T, S, \mathbb{R}^{m-s}, \pi_T) \overset{\kappa}{\cong} (N(S), S, \pi_{N(S)})$.*

$$\begin{tikzcd}
T \arrow[r, "\kappa"] \arrow[dr, "\pi_T"'] & N(S) \arrow[d, "\pi_{N(S)}"] \\
& S
\end{tikzcd}$$

### 4.1.3 Algebraic formula for the diagonal class

Let $M$ be a connected compact oriented smooth manifold of dimension $m$ equipped with the Riemannian metric. By definition, the diagonal class is the Poincaré dual of the submanifold $\Delta(M)$ in $M \times M$. However in order to obtain its algebraic formula it is needed to represent it under another form. Therefore in this subsection we are going to explain what a Poincaré dual is, give the Poincaré dual of $\Delta(M)$ regarded as a submanifold of $M \times M$, and show that it is equal to $D_M$ which is similar to the Thom class of $M \times M$. Moreover, we are going to explain how the diagonal class of $M$ is equal to the image of $1_{\mathbb{R}}$ via the copairing $\gamma$ when the dimension of the manifold $M$ is even.

With the notice that from now on when we say tubular neighborhood $T$ we refer to the fiber bundle, let us give an insight of the Poincaré dual of a submanifold.

**Definition 4.1.8.** *(?) Let $S$ be a closed oriented submanifold of dimension $k$ of $M$. Let $i : S \longrightarrow M$ be the inclusion of $S$ in $M$.*
*The Poincaré dual $\eta_S$ is the unique cohomology class in $H^{m-k}(M)$ satisfying*

$$\int_S i^*\omega = \int_M \omega \wedge \eta_S$$

*for any $\omega \in H_c^k(M)$.*

From the Theorem **??**, we have $T \overset{\kappa}{\cong} N(S)$ where $N(S)$ is the normal bundle over $S$ and $T$ is the tubular neighborhood of $S$ in $M$. So we have the Thom class of $N(S)$ denoted $Th_{N(S)}$. Therefore, $\kappa^*(Th_{N(S)}))$ is similar to the "Thom class" on $T$. And this result leads us to another formulation of the Poncaré dual of $S$.

**Theorem 4.1.9.** *(?) Let $S$ be a closed oriented submanifold of dimension $k$ of $M$. Let $j : T \longrightarrow M$ be the inclusion map.*
*The Poincaré dual of $S$ might be expressed as*

$$\eta_S = j_\bullet(\kappa^*(Th_{N(S)}) \wedge 1_{H^0(T)}) = j_\bullet(\kappa^*(Th_{N(S)}))$$

For a smooth manifold $M$, a smooth section of the vector bundle $(E, M, \pi_E)$ is basically a smooth map $s$ such that $\pi_E \circ s = id_M$. And it's called zero section when its image at $p$ is the zero vector of $\pi^{-1}(p)$ for all $p$ in $M$.
Those informations and Stokes theorem are needed for the proof of the above theorem. The Stokes theorem goes as follows:

**Theorem 4.1.10** (Stokes theorem)**.** *(**?**) Let $M$ be an oriented smooth manifold of dimension $m$. Let $\omega$ be a compactly supported $(m-1)-$differential form on $M$. If $\partial M$ is given the induced orientation, then*

$$\int_M d\omega = \int_{\partial M} \omega$$

*Proof.* (Theorem**??**)
From Definition **??**, $\eta_S$ is unique. Therefore to prove this equality, we need to check that for any compactly supported $k-$form $\omega$ on $M$ and $i : S \longrightarrow T$ the inclusion regarded as the zero section of the bundle $(T, S, \mathbb{R}^{m-s}, \pi_T)$ we have $\int_M \omega \wedge j_\bullet(\kappa^*(Th_{N(S)})) = \int_S i^*\omega$.
From the definition of a section we have $\pi_T \circ i = id_S$, then $(\pi_T)^*$ and $i^*$ are inverse isomorphisms in cohomology. Therefore $\omega = (\pi_T)^* i^*\omega + d\tau$. where $\tau \in \Omega^{k-1}(M)$. From the definition of $j_\bullet$, we may notice that $j_\bullet(\kappa^*(Th_{N(S)}))$ has support in $T$. Therefore

$$\begin{aligned}
\int_M \omega \wedge j_\bullet(\kappa^*(Th_{N(S)}) &= \int_T \omega \wedge \kappa^*(Th_{N(S)}) \\
&= \int_T ((\pi_T)^* i^*\omega + d\tau) \wedge \kappa^*(Th_{N(S)}) \\
&= \int_T ((\pi_T)^* i^*\omega) \wedge \kappa^*(Th_{N(S)}) \\
&\mathit{since} \int_T (d\tau) \wedge \kappa^*(Th_{N(S)}) = \int_T d(\tau \wedge \kappa^*(Th_{N(S)}) = 0 \\
&\text{(Stokes' theorem and the fact that the forms get zero at the boundary )} \\
&= \int_S i^*\omega \wedge (\pi_T)_* \kappa^*(Th_{N(S)}) \text{ by Proposition ??} \\
&= \int_S i^*\omega
\end{aligned}$$

□

Since now we have the two formulations of the Poincaré dual of a closed oriented submanifold $S$ of $M$, let us move to the construction of $D_M$.

The first step for that is to use the Theorem **??** by changing our $M$ into $M \times M$ and our $S$ into $M$. As we have stated at the beginning, our manifold $M$ is compact. Therefore, $M \times M$ is compact as well. The fact that $M$ might be considered as a compact submanifold of $M \times M$ leads us to a bundle isomorphism $(N(M), \Delta(M), \pi_{(N(M))}) \overset{\kappa}{\cong} (T, \Delta(M), \pi_T)$ where $T$ is a tubular neighborhood of $\Delta(M)$ in $M \times M$.
By applying the Theorem **??** we have the bundle isomorphisms

$$(T, \Delta(M), \pi_T) \overset{\kappa}{\cong} (N(M), \Delta(M), \pi_{N(M)}) \tag{4.1}$$

$$(N(M), \Delta(M), \pi_{N(M)}) \overset{\lambda}{\cong} (TM, M, \pi_{TM}) \tag{4.2}$$

And combined with the extension map

$$j_\bullet : H_c^m(T) \longrightarrow H^m(M \times M)$$

of $j : T \longrightarrow M \times M$ ( Chapter **??**), we may now define $D_M$.
So from the tangent bundle of $M$ we obtain the Thom class $Th_{TM} \in H^m(TM)$.
Then the Theorem **??** insures that $(\lambda^{-1})^*(Th_{TM}) \in H^m(N(M))$. And by using the Theorem **??** we obtain $\kappa^*(((\lambda^{-1})^*(Th_{TM})) \in H^m(T)$.
Finally, $D_M \in H^m(M \times M)$ is defined to be

$$D_M := j_\bullet(\kappa^*((\lambda^{-1})^*(Th_{TM})).$$

Notice that the form $D_M$ here is no longer a Thom class but only a differential form on $M \times M$.
After seeing the expression of $D_M$ and the Poincaré dual of a closed submanifold of $M$, we may conclude that:

**Theorem 4.1.11.** *Let $M$ be a compact oriented smooth manifold of dimension $m$. Then $D_M$ is exactly the diagonal class i.e.*

$$\eta_{\Delta(M)} = D_M.$$

*Proof.* By taking $S$ in Theorem **??** to be $\Delta(M)$ we have $\eta_{\Delta(M)} = j_\bullet(\kappa^*(Th_{N(\Delta(M))}))$.
But in the previous method, we have

$$D_M := j_\bullet(\kappa^*(((\lambda^{-1})^*(Th_{TM}))).$$

So it' s necessary to check that $Th_{N(\Delta(M))} = (\lambda^{-1})^*(Th_{TM})$. If we have a look at the following diagram,

$$\begin{tikzcd} TM \arrow[r, "\lambda", "\cong"'] \arrow[d, "\pi_{TM}"'] & N(M) \arrow[d, "\pi_{N(M)}"] \\ M \arrow[r, "\cong", "\Delta"'] & \Delta(M) \end{tikzcd}$$

we see that it commutes and it induces the commutative diagram

$$\begin{tikzcd} H_c^*(TM) \arrow[r, "(\lambda^{-1})^*"] \arrow[d, "(\pi_{TM})_*"'] & H_c^*(N(M)) \arrow[d, "(\pi_{N(M)})_*"] \\ M \arrow[r, "\Delta^*"'] & \Delta(M) \end{tikzcd}$$

which says:

$$(\Delta)^*((\pi_{N(M)})_*((\lambda^{-1})^*)(Th_{TM})) = (\pi_{TM})_*(Th_{TM}) = 1_{H^0(M)}.$$

Since $(\Delta)^*$ is an isomorphism, therefore $(\pi_{N(M)})_*((\lambda^{-1})^*(Th_{TM})) = 1_{H^0(M)}$ i.e. $(\lambda^{-1})^*(Th_{TM}) = Th_{N(M)}$.

□

As we see here, the formula of the diagonal class that we obtained is still not algebraic. In order to get to the explicit computation we first need the notion of Künneth isomorphism. So as follows is that theorem:

**Theorem 4.1.12** (Künneth). *(?) Let P and Q be two manifolds. Therefore*

$$\begin{aligned} K : H^*(P) \otimes H^*(Q) &\longrightarrow H^*(P \times Q) \\ \omega \otimes \alpha &\longmapsto p_1^*\omega \wedge p_2^*\alpha \end{aligned}$$

*such that*

$$\begin{aligned} p_1 : P \times Q &\longrightarrow P \\ (x, y) &\longmapsto x \\ p_2 : P \times Q &\longrightarrow Q \\ (x, y) &\longmapsto y \end{aligned}$$

*is an isomorphism.*

Our goal here is to write the diagonal class into an algebraic formula.
From the Künneth formula we may represent the diagonal class as a form in $H^*(M) \otimes H^*(M)$ that we still call diagonal class and will represent as $\hat{D}_M$.

As a result from the section before, the pairing $\beta$ is nondegenerate. Since $H^*(M) = \oplus_{k\leq m} H^k(M)$ and $H^k(M)$ is finite dimensional for all $k \leq m$, $H^*(M)$ is also finite dimensional as an algebra. So we may think of its basis and its dual basis.
Let $\{b_i\}_i$ be a basis for $H^*(M)$ and $\{b_i^\#\}_i$ be its dual basis i.e. $\int_M b_i \wedge b_j^\# = \delta_{i,j}$. In particular if $b_i \in H^k(M)$ then $b_i^\# \in H^{m-k}(M)$.
According to Milnor and Stasheff (**?**) $\hat{D}_M = K^{-1}(D_M)$ has an algebraic formula that is presented by the following theorem:

**Theorem 4.1.13.** *(**?**) The diagonal class $\hat{D}_M \in H^*(M) \otimes H^*(M)$ is given by*

$$\hat{D}_M = \sum_{i=1}^{r} (-1)^{\deg b_i} b_i \otimes b_i^\#$$

*where $r$ is the dimension of $H^*(M)$ as a vector space and* $\deg b_i$ *is the degree of the cohomology where $b_i$ lives.*

*Proof.* The aim of the proof is first to give the algebraic formula for $D_M$ in terms of $b_i$ and $b_i^\#$, then follows that of $\hat{D}_M$.
Notice that this proof is inspired from that of Milnor and Stasheff(**?**).
First of all, let us remind ourselves that if two maps are homotopic map, then their pullback are equal.
Let

$$\begin{aligned} p_1 : M \times M &\longrightarrow M \\ (x, y) &\longmapsto x \\ p_2 : M \times M &\longrightarrow M \\ (x, y) &\longmapsto y \end{aligned}$$

Since $p_1$ and $p_2$ are homotopic (they differ by rotation), they induce the same pullback maps i.e. $p_1^* = p_2^*$. Therefore for $\omega \in H^k(\Delta(M))$ for a fixed $k$, we have

$$p_1^*(\omega) \wedge D_M = p_2^*(\omega) \wedge D_M$$

Hence we have

$$(p_1)_*(p_1^*(\omega) \wedge D_M) = (p_1)_*(p_2^*(\omega) \wedge D_M)$$

and from the projection formula **??**, the left hand side gives us

$$\omega \wedge (p_1)_*(D_M) = \omega.$$

As an illustration from the Künneth isomorphism **??** we may write

$$D_M = \sum_{i=1}^{r} p_1^*(b_i) \wedge p_2^*(c_i), \deg b_i + \deg c_i = m$$

where the $c_i$s are certain forms on $M$. Our goal here is to write them in terms of the $b_i^\#$. Notice that for $\sigma \in H^*(\Delta(M))$

$$(p_1)_* p_2^* \sigma = \begin{cases} \int_M \sigma & \deg \sigma = m \\ 0 & otherwise \end{cases}$$

The right hand side gives us

$$\begin{aligned} (p_1)_*(p_2^*(\omega) \wedge D_M) &= \sum_{i=1}^{r} (p_1)_*(p_2^*(\omega) \wedge p_1^*(b_i) \wedge p_2^*(c_i)) \\ &= \sum_{i=1}^{r} (-1)^{\deg \omega \deg b_i} (p_1)_*(p_1^*(b_i) \wedge p_2^*(\omega) \wedge p_2^*(c_i)) \\ &= \sum_{i=1}^{r} (-1)^{\deg \omega \deg b_i} b_i \wedge (p_1)_*(p_2^*(\omega \wedge c_i)) \\ &= \sum_{i=1}^{r} (-1)^{\deg \omega \deg b_i} (\int_M \omega \wedge c_i) b_i \end{aligned}$$

Therefore,

$$\omega = \sum_{i=1}^{r} (-1)^{\deg \omega \deg b_i} \beta(\omega, c_i) b_i$$

By taking $\omega = b_j$ we have

$$b_j = \sum_{i=1}^{r} (-1)^{\deg b_j \deg b_i} \beta(b_j, c_i) b_i$$

So

$$(-1)^{\deg b_j \deg b_i} \beta(b_j, c_i) = \delta_{ji}$$

i.e.

$$\beta(b_j, c_i) = (-1)^{\deg b_j \deg b_i} \delta_{ji}$$

In particular when $i = j$ we have

$$\beta(b_j, c_j) = (-1)^{(\deg b_j)^2}$$

Thus

- if $\deg b_j$ is even, then $\beta(b_j, c_j) = 1$
- if $\deg b_j$ is odd, then $\beta(b_j, c_j) = -1$

Therefore $b_j^{\#} = (-1)^{\deg b_j} c_j$ i.e. $c_j = (-1)^{\deg b_j} b_j^{\#}$ which leads us to the algebraic formula of $D_M$:

$$\begin{aligned} D_M &= \sum_{i=1}^{r} p_1^*(b_i) \wedge p_2^*((-1)^{\deg b_i} b_i^{\#}) \\ &= \sum_{i=1}^{r} (-1)^{\deg b_i} p_1^*(b_i) \wedge p_2^*(b_i^{\#}) \end{aligned}$$

And the Künneth formula gives us that

$$\hat{D}_M = \sum_{i=1}^{r} (-1)^{\deg b_i} b_i \otimes b_i^{\#}$$

□

Furthermore, this brings us to new informations on the diagonal class which is given by:

**Lemma 4.1.14.** *If $M$ is of even dimension, then the diagonal class is equal to the image of the unit in $\mathbb{R}$ via the copairing i.e.*

$$\hat{D}_M = \gamma(1_{\mathbb{R}}) \textit{ when } \dim M \textit{ is even}.$$

*Proof.* Recall from Chapter 4 that

$$\gamma(1_{\mathbb{R}}) = \sum_{i=1}^{r} b_i^{\#} \otimes b_i$$

So our aim here is to know the condition on $\alpha_i$ such that

$$\gamma(1_{\mathbb{R}}) = \sum_i \alpha_i b_i \otimes b_i^{\#}$$

To compute that, we use the fact that $\gamma$ must satisfy the snake relations.

$$\begin{array}{ccccc} \mathbb{R} \otimes H^*(M) & \xrightarrow{\gamma \otimes id} & H^*(M) \otimes H^*(M) \otimes H^*(M) & \xrightarrow{id \otimes \beta} & H^*(M) \otimes \mathbb{R} \\ 1_{\mathbb{R}} \otimes b_i & \longmapsto & \sum_j \alpha_j b_j \otimes b_j^{\#} \otimes b_i & \longmapsto & \alpha_j (-1)^{\deg b_i \deg b_j^{\#}} b_i \int_M b_i \wedge b_j^{\#} \otimes 1_{\mathbb{R}} \end{array}$$

Therefore, in order to get

$$b_i = \alpha_i(-1)^{\deg b_i \deg b_i^\#} b_i$$

we need

$$\alpha_i = (-1)^{\deg b_i \deg b_i^\#}.$$

Until now we have only used the first snake, we need to check if it satisfies the second snake.

$$\begin{array}{ccccc} H^*(M)\otimes\mathbb{R} & \xrightarrow{id\otimes\gamma} & H^*(M)\otimes H^*(M)\otimes H^*(M) & \xrightarrow{\beta\otimes id} & \mathbb{R}\otimes H^*(M) \\ b_i^\# \otimes 1_\mathbb{R} & \longmapsto & \sum_i (-1)^{\deg b_j \deg b_j^\#} b_i^\# \otimes b_j \otimes b_j^\# & \longmapsto & 1_\mathbb{R}\otimes b_i^\# = b_i^\# \end{array}$$

Thus, $\gamma(1_\mathbb{R}) = \sum_i (-1)^{\deg b_i \deg b_i^\#} b_i \otimes b_i^\#$.
By looking at the following table:

| | $\deg b_i$ | $\deg b_i^\#$ | $(-1)^{\deg b_i \deg b_i^\#}$ | $(-1)^{\deg b_i}$ |
|---|---|---|---|---|
| $\dim M = even$ | *even* | *even* | 1 | 1 |
| | *odd* | *odd* | $-1$ | $-1$ |
| $\dim M = odd$ | *even* | *odd* | 1 | 1 |
| | *odd* | *even* | 1 | $-1$ |

we may conclude that $(-1)^{\deg b_i \deg b_i^\#} = (-1)^{\deg b_i}$ when the dimension of $M$ is even, so we have $\hat{D}_M = \gamma(1_\mathbb{R})$. ☐

**Lemma 4.1.15.** *If $M$ is of odd dimension, then the diagonal class is not equal to the image of the unit in $\mathbb{R}$ via the copairing i.e.*

$$\hat{D}_M \neq \gamma(1_\mathbb{R}) \text{ when } \dim M \text{ is odd.}$$

*Proof.* From the table in the previous proof (proof of Lemma **??**), one may think that $(-1)^{\deg b_i \deg b_i^\#} \neq (-1)^{\deg b_i}$ but the sum might be the same when $\dim M$ is odd. In fact, that is impossible.
In general for any connected compact oriented manifold $M$ of dimension $m$, since we know that $(H^k(M))^* = H^{m-k}(M)$, we have:

| | $H^0(M)$ | $H^1(M)$ | $\cdots$ | $H^m(M)$ |
|---|---|---|---|---|
| basis | 1 | $\cdots$ | $\cdots$ | $\alpha$ |
| dual basis | $\alpha$ | $\cdots$ | $\cdots$ | 1 |

So

$$\begin{aligned}\gamma(1_{\mathbb{R}}) &= \sum_i (-1)^{\deg b_i \deg b_i^{\#}} b_i \otimes b_i^{\#} \\ &= 1 \otimes \alpha + \sum_{b_i \neq 1, \alpha} (-1)^{\deg b_i \deg b_i^{\#}} b_i \wedge b_i^{\#} + \alpha \otimes 1 \\ &= 1 \otimes \alpha + \alpha \otimes 1 + \sum_i (-1)^{\deg b_i \deg b_i^{\#}} b_i \wedge b_i^{\#}\end{aligned}$$

and

$$\begin{aligned}\hat{D}_M &= \sum_i (-1)^{\deg b_i} b_i \otimes b_i^{\#} \\ &= 1 \otimes \alpha + \sum_{b_i \neq 1, \alpha} (-1)^{\deg b_i} b_i \otimes b_i^{\#} + (-1)^m \alpha \otimes 1 \\ &= 1 \otimes \alpha + (-1)^m \alpha \otimes 1 + \sum_{b_i \neq 1, \alpha} (-1)^{\deg b_i} b_i \wedge b_i^{\#} \\ &= 1 \otimes \alpha - \alpha \otimes 1 + \sum_{b_i \neq 1, \alpha} (-1)^{\deg b_i} b_i \wedge b_i^{\#} \text{ dimension of } M \text{ odd}\end{aligned}$$

Therefore

$$\begin{aligned}\gamma(1_{\mathbb{R}}) - \hat{D}_M &= 2\alpha \otimes 1 + \sum_{b_i \neq 1, \alpha} (-1)^{\deg b_i} [(-1)^{\deg b_i^{\#}} - 1] b_i \otimes b_i^{\#} \\ &\neq 0\end{aligned}$$

□

## 4.2 Euler class of $M$

Let $M$ be a connected compact oriented smooth manifold of dimension $m$. Now that we are done with the diagonal class, let us focus more on the Euler class of $M$. Our aim in this subsection is to describe how to get its algebraic definition and to prove that when the dimension of the manifold $M$ is even the Euler class of $M$ is equal to the handle operator of the de Rham cohomology ring $H^*(M)$. So let us start with a reminder of a what an Euler class is in general.

**Definition 4.2.1.** *(?) Let $M$ be a smooth oriented manifold.*
*Let $\xi = (E, M, F, \pi_E)$ be a vector bundle of rank $n$ over $M$.*
*Let $s : M \longrightarrow E$ be any section.*

*The **Euler class** $e(\xi) \in H^n(M)$ of $\xi$ is defined as*

$$e(\xi) = s^* Th_E$$

In our case, the manifold $M$ is compact, oriented, smooth, and of dimension $m$, and the Euler class $e_M \in H^m(M)$ is given by $e_M = s^*(Th_{TM})$. This is the topological definition of an Euler class. But as stated before we would like to give its algebraic definition. The following theorem is the first step to get there.

**Theorem 4.2.2.** *The Euler class of M is equal to the pullback of the diagonal class by the diagonal map $\Delta$ i.e.*

$$e_M = \Delta^*(D_M).$$

*Proof.* To prove this theorem we need to show that the following diagram commutes:

$$\begin{tikzcd}
H^*(TM) \arrow[r, "\lambda^*"] \arrow[drrr, "s^*"'] & H^*(N(M)) \arrow[r, "\kappa^*"] & H^*(T) \arrow[r, "j_\bullet"] & H^*(M \times M) \arrow[d, "\Delta^*"] \\
 & & & H^*(M)
\end{tikzcd}$$

By the definition of fiber bundles isomorphism in the Definition **??** and all the results from Chapter **??** we see that the following diagrams commute.

$$\begin{tikzcd}
TM \arrow[r, leftrightarrow, "\cong"] \arrow[d, shift right, "\pi_{TM}"'] & N(M) \arrow[r, leftrightarrow, "\cong"] \arrow[d, shift right, "\pi_{N(M)}"'] & T \arrow[d, shift right, "\pi_T"'] \\
M \arrow[r, leftrightarrow, "\cong"'] \arrow[u, shift right, "s_0"'] & \Delta(M) \arrow[r, leftrightarrow, "id"'] \arrow[u, shift right, "s_0"'] & \Delta(M) \arrow[u, shift right, "i"']
\end{tikzcd}$$

However, by considering the zero sections and by regarding the inclusion $i$ of $\Delta(M)$ into $T$ as the zero section of $T$, those commutative diagrams induce the commutativity of the following diagram:

$$\begin{tikzcd}
H^*(TM) \arrow[r, "\lambda^*"] \arrow[drr, "s^*"'] & H^*(N(M)) \arrow[r, "\kappa^*"] & H^*(T) \arrow[d, "i^*"] \\
 & & H^*(\Delta(M)) = H^*(M)
\end{tikzcd}$$

Since $i^*$ is an isomorphism, and by taking $\iota$ to be the inclusion of $\Delta(M)$ in $M \times M$, it's obvious to see that the following diagram commutes:

$$\begin{tikzcd} H^*(T) \arrow[r, "j_\bullet"] \arrow[d, "i^*"'] & H^*(M \times M) \arrow[dl, "\iota^*"] \\ H^*(M) & \end{tikzcd}$$

Thus, the first diagram commutes and $e_M = \Delta^*(D_M)$. □

However, from the Theorem **??** we have $D_M = K(\hat{D}_M)$ so the Euler class can be written as

$$e_M = \Delta^*(K(\hat{D}_M)).$$

And since we have the algebraic formula for the diagonal class $\hat{D}_M$, the algebraic formula for the Euler class might be defined.

**Theorem 4.2.3.** *The algebraic formula for the Euler class of $M$ is given by:*

$$e_M = \sum_{i=1}^{r} (-1)^{\deg b_i} b_i \wedge b_i^{\#}$$

*Proof.* When looking at the algebraic formula for the diagonal class $\hat{D}_M$, we realize that the expression of the Euler class here is the same as $\mu(\hat{D}_M)$. So if we are able to prove that the equality between $\mu$ and $\Delta^* \circ K$, then we are done. Actually, this equality resumes to the commutativity of the following diagram:

$$\begin{tikzcd} H^*(M) \otimes H^*(M) \arrow[r, "K"] \arrow[dr, "\mu"'] & H^*(M \times M) \arrow[d, "\Delta^*"] \\ & H^*(M) \end{tikzcd}$$

It is known that $M \cong \Delta(M)$, so their De Rham cohomology groups are isomorphic i.e. $H^k(M) \overset{\psi}{\cong} H^k(\Delta(M))$ for all $k \leq m$.
Consider the maps

$$\begin{aligned} p_1 : M \times M &\longrightarrow M \\ (x, y) &\longmapsto x \\ p_2 : M \times M &\longrightarrow M \\ (x, y) &\longmapsto y \end{aligned}$$

Then the definition of the composite map is given by

$$\begin{aligned} H^*(M) \otimes H^*(M) &\xrightarrow{K} H^*(M \times M) \xrightarrow{\Delta^*} H^*(M) \\ \omega_1 \otimes \omega_2 &\mapsto p_1^*(\omega_1) \wedge p_2^*(\omega_2) \mapsto \Delta^*(p_1^*(\omega_1) \wedge p_2^*(\omega_2)) \end{aligned}$$

Therefore, all we need to prove is

$$\Delta^*(p_1^*(\omega_1) \wedge p_2^*(\omega_2)) = \omega_1 \wedge \omega_2.$$

But that requires the explicit definition of $p_1^*$ and $p_2^*$. So let

$$\begin{aligned} p_i^* : H^k(M) &\longrightarrow H^k(M \times M) \\ \omega_i &\longmapsto \psi(\omega_i) \end{aligned}$$

for $i = 1, 2$.
Since $\Delta^* \restriction_{H^*(\Delta(M))}$ is an isomorphism and it's exactly the inverse of $\psi$. Thus

$$\Delta^*(p_1^*(\omega_1) \wedge p_2^*(\omega_2)) = \omega_1 \wedge \omega_2,$$

which means the commutativity of the diagram above and therefore the equality between $\mu$ and $\Delta^* \circ K$. And that result leads us to the algebraic expression of the Euler class of $M$.

□

We can now present the main conclusion of this thesis, namely the relationship between the handle element and the Euler class.

**Theorem 4.2.4.** *The Euler class of $M$ is equal to the handle element of $H^*(M)$ when the dimension of $M$ is even i.e.*

$$e_M = (\mu \circ \gamma)(1_{\mathbb{R}}) \textit{ when } \dim M \textit{ is even}.$$

*Moreover, if the dimension of $M$ is odd, then the Euler class of $M$ is not equal to the handle element of $H^*(M)$.*

*Proof.* Recall that

$$e_M = \sum_{i=1}^{r} (-1)^{\deg b_i} b_i \wedge b_i^\#$$

and

$$(\mu \circ \gamma)(1_{\mathbb{R}}) = \sum_i (-1)^{\deg b_i \deg b_i^\#} b_i \wedge b_i^\#$$

- Suppose that M is of even dimension

  The proof carries over verbatim as that of the lemma **??** by changing all the $\otimes$ into $\wedge$. Furthermore we have

$$\begin{aligned}\int_M (\mu \circ \gamma)(1_{\mathbb{R}}) &= \sum_{i=1}^{r} \int_M (-1)^{\deg b_i} b_i \wedge b_i^{\#} \\ &= \sum_{i=1}^{r} \int_M (-1)^{\deg b_i} \\ &= \sum_{i=0}^{m} (-1)^i \dim H^i(M) \\ &= \chi(M)\end{aligned}$$

  which is a known formula for the Euler characteristic of M.

- Supppose that M is of odd dimension

  Therefore , we get

$$\int_M (\mu \circ \gamma)(1_{\mathbb{R}}) = \sum_{i=0}^{m} \dim H^i(M)$$

  which is equal to the total (unsigned) dimension of the cohomology of M, and hence cannot be equal to the Euler characteristic of M.

$\square$

## 4.3 Examples

For more understanding let us look at some examples.

### 4.3.1 The circle $M = S^1$

Let's look at the simplest case where the manifold is $S^1$, the unit circle.

| | $H^0(S^1)$ | $H^1(S^1)$ |
|---|---|---|
| basis | $1$ | $\alpha$ |
| dual basis | $\alpha$ | $1$ |

The Euler class is given by:

$$\begin{aligned} e_{S^1} &= \sum_i (-1)^{\deg b_i} b_i \wedge b_i^{\#} \\ &= 1 \wedge \alpha - \alpha \wedge 1 \\ &= 0 \end{aligned}$$

and the handle element is given by:

$$\begin{aligned} (\mu \circ \gamma)(1_{\mathbb{R}}) &= \sum_i b_i^{\#} \wedge b_i \\ &= \alpha \wedge 1 + 1 \wedge \alpha \\ &= 2\alpha \end{aligned}$$

### 4.3.2 The torus $M = S^1 \times S^1$

For the case of the torus, we have the following table:

| | $H^0(M)$ | $H^1(M)$ | $H^2(M)$ |
|---|---|---|---|
| basis | $1$ | $\alpha, \beta$ | $\alpha \wedge \beta$ |
| dual basis | $\alpha \wedge \beta$ | $\beta, -\alpha$ | $1$ |

The Euler class is given by:

$$\begin{aligned} e_M &= (-1)^0 \alpha \wedge \beta + (-1)^1 \alpha \wedge \beta + (-1)^1 \beta \wedge (-\alpha) + \alpha \wedge \beta \\ &= 0 \end{aligned}$$

And the handle element is given by:

$$\begin{aligned} (\mu \circ \gamma)(1_{\mathbb{R}}) &= \alpha \wedge \beta + \beta \wedge \alpha - \alpha \wedge \beta + \alpha \wedge \beta \\ &= 0 \end{aligned}$$

### 4.3.3 A genus g surface $\Sigma_g$

Let $M$ be a genus $g$ surface. The basis and dual basis of its cohomology ring is given by the following table:

| | $H^0(M)$ | $H^1(M)$ | $H^2(M)$ |
|---|---|---|---|
| basis | $1$ | $\alpha_i, \beta_i$ | $\omega$ |
| dual basis | $\omega$ | $\beta_i, -\alpha_i$ | $1$ |

where $i \in \{1, \cdots, g\}$, $\omega = \alpha_1 \wedge \beta_1 = \cdots = \alpha_g \wedge \beta_g$ and $\int_M \omega = 1$.
The Euler class is given by

$$\begin{aligned}
e_M &= \sum_{i=1}^{g} (-1)^{\deg \alpha_i} \alpha_i \wedge \beta_i + \sum_{i=1}^{g} (-1)^{\deg \beta_i} \beta_i \wedge (-\alpha_i) + 2\omega \\
&= \sum_{i=1}^{g} (-1)^{\deg \alpha_i} \alpha_i \wedge \beta_i + \sum_{i=1}^{g} (-1)^{\deg \beta_i} \alpha_i \wedge \beta_i + 2\omega \\
&= 2 \sum_{i=1}^{g} (-1)^{\deg \alpha_i} \alpha_i \wedge \beta_i + 2\omega \ \text{ since } \alpha_i, \beta_i \in H^1(M) \\
&= (2 \sum_{i=1}^{g} (-1) + 2)\omega \text{ since } \deg \alpha_i = 1 \text{ and } \alpha_i \wedge \beta_i = \omega \\
&= (2 - 2g)\omega
\end{aligned}$$

and the Euler characteristic follows as

$$\begin{aligned}
\chi(M) &= (2 - 2g) \int_M \omega \\
&= 2 - 2g
\end{aligned}$$

## 4.4 Corrections and clarifications to literature

In summary we would like to point out the following corrections and refinements to the literature.

- The exercise in (**?**, pg 131, exercise 22) should be qualified to hold for even-dimensional manifolds:

  > Let $X$ be a compact connected orientable even-dimensional manifold of dimension $r$, and put $A = H^*(X)$. Show that the Euler class (top Chern class of the tangent bundle, $c_r(TX)$) is the handle element of $A$.

- Moreover, we point out that a refinement of this statement is true, namely:

  > Let $X$ be a compact connected orientable even-dimensional manifold of dimension $r$, and put $A = H^*(X)$. Then the diagonal class of $M$ (the avatar of the Thom class) is equal to the comultiplication of the unit of $\mathbb{R}$.

- The formula for the Euler class in Abrams (**?**, bottom of page 4),

$$e(X) = \sum_i e_i e_i^{\#} \tag{4.3}$$

seems to be a misquotation of (**?**, Theorem 11.11). It should read

$$e(X) = \sum_i (-1)^{\dim e_i} e_i e_i^{\#}. \tag{4.4}$$

Abrams does qualify his formula as holding for even-dimensional manifolds (first sentence on page 4). But even in that case, (**??**) is still incorrect. It is possible he meant to write

$$e(X) = \sum_i e_i^{\#} e_i \tag{4.5}$$

which holds for even-dimensional $M$.